AEROELASTIC STABILITY ANALYSIS OF AN AIRFOIL WITH STRUCTURAL NONLINEARITIES USING A STATE SPACE UNSTEADY AERODYNAMICS MODEL

H. S. Murty*
Rotorcraft Research Technology Center
Scientific Computation Research Center
Rensselaer Polytechnic Institute
Troy, NY 12180

Abstract

Aeroelastic stability analysis is carried out with a structural cubic nonlinearity in torsional stiffness, for a two-degree-of-freedom airfoil. This analysis was carried out using a state space formulation for the aerodynamic model. This formulation enables hard coupling between the aerodynamics and the structural dynamics. The stability of the system is determined by carrying out an eigenvalue analysis. This analysis was applied to the case of a NACA 64A006 airfoil at a freestream Mach number of 0.85. As limit cycle oscillations can occur prior to the flutter point for nonlinear spring stiffnesses, a time response analysis is carried out in order to determine system behaviour. The set of conditions considered in this study produced lower flutter speeds with a nonlinearity than the linear flutter speed.

Nomenclature

A	coefficients of indicial functions
b	semichord
C_N	lift coefficient
$C_{n\alpha}$	lift curve slope
C_M	moment coefficient
K	stiffness
K	ratio of nonlinear restoring moment to linear stiffness
L	lift
M	mach number
m	mass
q	pitch rate
r_α	radius of gyration about elastic axis
S	nondimensional time
u	displacement vector
x_α	distance between elastic axis to mass centre
α	angle of attack
β	compressibility factor
λ	eigenvalue
μ	airfoil mass ratio
ρ	air density
ϕ	indicial response
ω_h	uncoupled bending frequency
ω_α	uncoupled torsion frequency

1. Motivation

Stability analysis of structural dynamic systems using an eigenvalue formulation has been a standard procedure for many years. It can also be used to determine the aeroelastic stability of a system. However, the governing equations for transonic flows are nonlinear making an exact eigenvalue analysis difficult. In addition, the transonic flutter dip presents a significant drop in the flutter speed in the transonic speed range indicating the importance of flutter prediction at transonic speeds. Predictions of flutter for applications to rotorcraft and turbomachinery is complicated by the spanwise varying flow field encountered on a three dimensional blade. For fixed wing applications, an apriori set of flow conditions can be specified for the determination of the flutter stability boundary. For rotary wings, however, the range of flow fields encountered by the blade are variable and interference with rotor/fuselage interaction cannot be ignored. Aeroelasticians have carried out extensive efforts in the accurate prediction of flutter in turbomachinery components. Advanced turboprop (propfan) engines have created an elevated interest in the accurate flutter analysis of bladed disks.

Free-play nonlinearity in torsion was initially

---

*Research Associate, Member AIAA

investigated by Woolston, Runyan and Andrews[1] who found that limit cycle oscillations occurred at velocities much lower than the linear flutter speed for certain values of the initial pitch displacement.

The describing function method, which is derived from the method of harmonic balance for concentrated nonlinearities, was applied by Shen[2]. This method assumes a sinusoidal displacement and the subsequent load is represented by a Fourier series expansion. An effective stiffness is then defined as the ratio of the amplitude of the first term in the series and the displacement amplitude. This technique has been successfully used for freeplay type of nonlinearities for which the amplitude of the motion is larger than the amplitude of the freeplay. If the method of Krylov-Bogoliubov-Mitropolsky (KBM)[3] is used, then higher order harmonic terms can be included in the loads expansion for amplitudes of motion which are smaller than the amplitude of freeplay. The KBM linearization approach is based on the assumption that a nonlinear element can be approximately replaced by a liniear element with equivalent internal energy when the element is activated at an equivalent amplitude level.

Laurenson and Trn[4] also used this method for a three-dimensional wing with two root rotational nonlinear springs and noted that the presence of a freeplay nonlinearity tends to cause the effective system stiffness to be less than that of the linear system and this relationship is a function of the amplitude of motion.

McIntosh, Reed and Rodden[5] considered an airfoil with bilinear structural restraints in bending and torsion. The aerodynamic loads were predicted using Wagner's function for incompressible flow. Limit cycle oscillations (LCOs) were obtained for velocities well below the linear flutter boundary. Lee[6] observed similar behaviour.

Ueda and Dowell[7] used a variation of the describing function method which takes into account the first fundamental harmonic of the nonlinear oscillatory motion, in order to include the transonic aerodynamic nonlinearity. Limit cycle oscillations in transonic flows were also studied by Dowell and Ilgamov[8].

Yang and Zhao[9] also implemented the harmonic balance method for airfoils in incompressible flow for which Theodorsen's function can be used for harmonic motion. These authors obtained two stable limit cycles of different amplitudes for certain values of velocity.

For realistic aircraft applications, Lee and Tron[10] carried out flutter analysis for the CF-18(F/A-18) with structural nonlinearities using the describing function approach. A bilinear spring was placed at the wing-fold hinge and at the leading-edge flap hinge where free play was introduced. LCOs were detected at speeds lower than the flutter speed, even though flutter modes and frequencies remain unchanged at various values of hinge stiffness.

The influence of structural nonlinearities on the transonic aeroelastic behaviour of an airfoil was first considered by Tjatra et al[11]. The nonlinearity consisted of a torsional spring with preload and freeplay. Computations were carried out for a NACA 64A006 airfoil at varying Mach numbers using a transonic, small-disturbance code. A second-order asymptotic expansion technique was developed to account for the contributions of the higher harmonic terms in the effective stiffness. This analysis predicted a lower flutter speed for a preload. In the case of freeplay of course, the results depended on the ratio of the amplitude of motion to freeplay width.

It is also true that chaos could be exhibited by the system for certain values of airfoil parameters as suggested by Hauenstein et al[12], who carried out experimental and theoretical investigations for a rigid wing with freeplay nonlinearities in torsion and bending. The aerodynamics was predicted with an unsteady, subsonic, doublet-lattice technique and a significant aeroelastic response was exhibited at airspeeds lower than that at which neutral stability was predicted.

Tang and Dowell[13] considered a nonrotating helicopter blade with a NACA 0012 section and a pitch freeplay structural nonlinearity. Their study found that chaotic motion can be obtained when the motion of the system is near the flutter boundary or when stall occurs. A bifurcation based on limit cycle flutter speed separated the behaviour of the system into that dominated by the structural nonlinearity and that dominated by aerodynamic stall. Both flapping and pitching motions were investigated.

A more recent study carried out by Tanrikulu et al[14] analysed a multiple-degree-of-freedom nonlinear structure using the describing function formulation for symmetrical nonlinearities to determine the forced harmonic response. Several types of nonlinear stiffnesses were analysed, such as cubic, piecewise linear and coulomb friction.

The aeroelastic response of an airfoil with bilinear and cubic structural nonlinearities in the torsional spring was analysed by Price et al[15]. Incompressible flow was assumed and the analysis was carried out using a time response as well as the describing function technique. The presence of LCOs at speeds lower than the flutter speed were indicated.

It should be noted that accurate modelling of unsteady aerodynamics plays an important role in the aeroelastic design of both fixed-wing and rotary wing aircraft. However, this accurate determination is time consuming for aeroelastic analysis. Therefore, attempts to obtain an approximate evaluation of aerodynamic coefficients have been carried out. In particular, the indicial response method has been used successfully in conjunction with the superposition principle.

In order to simultaneously solve the aerodynamic equations with the structural equations of motion, a convenient formulation was given by Leishmann and Nguyan[16] where the set of aerodynamic and structural equations are expressed in a state space formulation. This formulation allows the system to be expressed as a set of first order ordinary differential equations. The stability of the aeroelastic system can then be determined by an eigenvalue analysis.

In this paper, the aeroelastic stability analysis was carried out with a structural nonlinearity in torsional stiffness modelled as a concentrated nonlinearity. Brietbach[17] has given a review of possible types of nonlinearities and the type discussed in this paper is a cubic nonlinearity. This type of nonlinearity comes about in cases of loose or worn control surface hinges. Recently Alighanbari and Price[18] discussed the flutter response of an airfoil with a structural nonlinearity and obtained interesting types of bifurcations below the linear flutter velocity as well as chaotic oscillations. For a cubic nonlinearity, it is possible to carry out a Lyapunov exponent calculation in order to verify chaos as carried out in Reference 18. However, to date, all such studies have been carried out in incompressible flow using Wagner's functions to predict the aerodynamics coefficients. In the present paper, the flow is transonic and the aerodynamics is computed using a state-space formulation. In addition, an unsteady, full potential code is used to verify the systems time response behaviour.

2. Aeroelastic Stability Analysis

2.1 State Space Formulation

Crouse and Leishman[19] developed a state space formulation for unsteady, transonic, two-dimensional flow. This aerodynamic system consists of a set of first order differential equations. The inputs to this system consist of the angle of attack, pitch rate and freestream Mach number. The outputs are the unsteady lift and moment.

A general aerodynamic system is defined as follows:

$$\dot{x} = Ax + Bu \qquad (1)$$

where x is a vector of length n and output equations,

$$y = Cx + Du \qquad (2)$$

where $u = u_i$, $i = 1, 2, \ldots, m$. In this case, u is a vector consisting of angle of attack and pitch rate and y is a vector consisting of the lift and moment and x_i, $i = 1, \ldots, n$ are the aerodynamic state variables. That is, x consists of the minimum amount of information required to fully determine the system. State space aerodynamic formulations have been widely used in the past for incompressible flow by reformulating Wagner's function (Edwards et al[20]). However, for rotary wing aeroelastic problems and most fixed wing applications, incompressible flow aeroelastic analysis is of limited use, and therefore extensions to include Mach number effects are of great value. Leishman[21] first attempted an extension of this formulation to subsonic flows using Laplace transforms of indicial response functions which can be represented by exponential approximations. This extension was carried out by decomposing the indicial response into a physical meaningful noncirculatory part and the circulatory part corresponding to the wake effects.

For example, the indicial normal force response to a step change in angle of attack is given as

$$\frac{C_N(S)}{\alpha} = \frac{4}{M}\phi^I_\alpha + C_{N\alpha}\phi^C_\alpha \quad (3)$$

where $C_{n\alpha}$ is the lift curve slope generally given as $2\pi/\beta$ where β is the Prandtl-Glauert modifications for compressibility. However, it is known that transonic aerodynamic lift curve slopes differ from this value significantly as demonstrated by Murty[22] and Murty and Johnston[23]. Hence the indicial response for the circulation loads and moments have all been modified by values for the lift curve slope given by an unsteady, full potential computational fluid dynamics code[22]. Included in this model is a modification for the location of the aerodynamic centre which can be determined for experimental airfoil data.

The complete set of aerodynamic equations form an eight state model composed of circulatory and noncirculatory loads and moments and the inputs to the system are the angle of attack and pitch rate, with outputs lift and moment. Details of this state space formulation are given by Leishman and Crouse[24]

$$\dot{x} = Ax + B\begin{bmatrix}\alpha \\ y\end{bmatrix} \quad (4)$$

$$\begin{bmatrix}C_N \\ C_M\end{bmatrix} = Cx + D\begin{bmatrix}\alpha \\ q\end{bmatrix} \quad (5)$$

2.2 Two Degree of Freedom Aeroelasticity

Flutter calculations can be carried out in either the time domain or frequency domain. Generally, analysis in the time domain is used for arbitrary unsteady motion whereas analysis in the frequency domain requires harmonic blade motion. Analysis of the unsteady flow field around a harmonically rotating blade is easier to determine than that undergoing arbitrary motion.

Figure 1 shows an airfoil undergoing motion in two degrees of freedom. The equations of motion for this system can be written as follows:

$$\begin{aligned} &\ddot{\xi} + x_\alpha\ddot{\alpha} + \zeta_h\dot{\xi}\omega + \omega_h^2\xi = -L/mb \\ &x_\alpha\ddot{\xi} + r_\alpha^2\ddot{\alpha} + \zeta_\alpha\dot{\alpha}\omega \\ &+ K(\alpha)/mb^2 = M/mb^2 \end{aligned} \quad (6)$$

where K(α) is the nonlinear spring stiffness. In matrix form,

$$\begin{bmatrix}1 & x_\alpha \\ x_\alpha & r_\alpha^2\end{bmatrix}\begin{bmatrix}\ddot{\xi} \\ \ddot{\alpha}\end{bmatrix} + \begin{bmatrix}\zeta_h\omega & 0 \\ 0 & \zeta_\alpha\omega\end{bmatrix}\begin{bmatrix}\dot{\xi} \\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}\omega_h^2 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\xi \\ \alpha\end{bmatrix} + \begin{bmatrix}0 \\ K(\alpha)/mb^2\end{bmatrix} = \begin{bmatrix}-qc/mb\ C_L \\ qc^2/mb^2\ C_M\end{bmatrix} \quad (7)$$

From the state space formulation, the lift and moment coefficient can be written in the following form,

$$\begin{bmatrix}C_L \\ C_M\end{bmatrix} = Cx + Dz \quad (8)$$

where x is a state space vector and z is $[\alpha\ q]^T$. Therefore,

$$\begin{bmatrix}\frac{-qc}{mb}\ C_L \\ \frac{qc^2}{mb^2}\ C_M\end{bmatrix} = C'x + \begin{bmatrix}D_1' & D_2'\end{bmatrix} \quad (9)$$

The nonlinear torsional spring restoring moment $\tilde{K}$ is modelled as a cubic nonlinearity(Fig. 2)

$$\tilde{K}(\alpha) = \beta_0 + \beta_1\alpha + \beta_2\alpha^2 + \beta_3\alpha^3 \quad (10)$$

For a nonlinear spring $\tilde{K}(\alpha) = \alpha$ and in order to solve for the aeroelastic stability, the describing function method is used to obtain the equivalent linear system. In this study, we assume there is no preload. Therefore the airfoil pitch motion is of the form

$$\alpha(\tau) = \alpha_1 \sin\omega\tau \qquad (11)$$

The describing function is therefore given by

$$\tilde{K}(\alpha) = k_\alpha(1 + \epsilon B_1 + \epsilon^2 B_2 \ldots) \qquad (12)$$

and following the analysis given by Bogoliubov and Mitropolsky[3],

$$\tilde{K}(\alpha) = k_\alpha\, \alpha_1\, (\beta_1 + \frac{3}{4}\beta_3\, \alpha_1^2) \qquad (13)$$

where α_1 is the amplitude of pitching oscillation.

This linearization represents the effective stiffness obtained from an asymptotic expansion as a function of amplitude and no preload as shown in Fig. 3.

Therefore, the set of aerodynamic and structural equations of motion can be written in the following form,

$$\begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ D_1'-k & D_2'-g & C' \\ B' & & A \end{bmatrix} \begin{bmatrix} z \\ x \end{bmatrix} \qquad (14)$$

Letting $\tilde{q} = [z\ x]^T$,

$$\tilde{q}_t = \begin{bmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & I & 0 \\ D_1'-k & D_2'-g & C' \\ B' & & A \end{bmatrix} \tilde{q} \qquad (15)$$

or

$$\tilde{q}_t = Q(\tilde{q}) \qquad (16)$$

The matrix Q is 12 x 12 and it is important to note that the state space formulation of Leishman had to be modified in order to take into account the nonlinear stiffness. The eigenvalues of the matrix associated with Q can be used to determine aeroelastic stability of the system. The eigenvalues are complex numbers

$$\lambda_k = \sigma_k + i\omega_k \qquad (17)$$

For any $\sigma_k > 0$, the system is unstable. Alternatively, a damping ratio can be computed for each mode and this is given by

$$\zeta_k = \frac{-\sigma_k}{\sqrt{\sigma_k^2 + \omega_k^2}} \qquad (18)$$

and if any damping ratio is negative the system is unstable.

In addition, the system is capable of undergoing limit cycle oscillations at speeds below the flutter speed as demonstrated by other authors. Hence it is advisable to examine the time response analysis of the system using a more exact computational fluid dynamics code for the conditions considered here. An unsteady, full potential code[22] is used to carry out this time response computations. The Wilson-θ method[25] was used for the time integration as given in Reference 22.

$$\dot{u}(\tau) \cong \dot{u}(\tau-\Delta\tau) + \frac{1}{2}\Delta\tau\ddot{u}(\tau-\Delta\tau) + \frac{1}{2}\Delta\tau\ddot{u}(\tau) \qquad (19)$$

and

$$u(\tau) = u(\tau-\Delta\tau) + \Delta\tau\dot{u}(\tau-\Delta\tau) + \frac{1}{3}\Delta\tau^2\ddot{u}(\tau-\Delta\tau) + \frac{1}{6}\Delta\tau^2\ddot{u}(\tau) \qquad (20)$$

Substituting into the aeroelastic equations

$$[M][\ddot{u}] + [C][\dot{u}] + [K][u] = [F] \qquad (21)$$

gives

$$\left(M] + [C]\frac{\Delta\tau}{2} + [K]\frac{\Delta\tau^2}{6} \right) \ddot{u}(\tau + \dot{u}(\tau-\Delta\tau)([C] + [K]\Delta\tau) + \ddot{u}(\tau-\Delta\tau)\left(C\frac{\Delta\tau}{2} + K\frac{\Delta\tau^2}{3} \right) + [K]u(\tau-\Delta\tau) = [F] \tag{22}$$

Inverting,

$$\ddot{u}(\tau) = \left([M] + [C]\frac{\Delta\tau}{2} + [K]\frac{\Delta\tau^2}{6} \right)^{-1} [F] - [K]u(\tau-\Delta\tau) - [C\frac{\Delta\tau}{2} + K\frac{\Delta\tau^2}{3}]\ddot{u}(\tau-\Delta\tau) - [C + K\Delta\tau]\dot{u}(\tau-\Delta\tau) \tag{23}$$

which determines the displacements at time τ based on information at the previous time step.

3. Results and Discussion

The state space model was used in this investigation for transonic flows with airfoils having cubic structural nonlinearities. Lieshman's formulation was therefore modified in order to incorporate nonlinearities. The state equations governing the unsteady aerodynamic behaviour of the airfoil have been combined with the structural equations of motion to form a system of first order differential equations. These differential equations were represented as a matrix equation on which standard eigenvalue procedures were used.

An application of this analysis was carried out with a NACA 64A006 airfoil at a freestream Mach number of 0.85 and the following mass and stiffness parameters: $x_\alpha = 0.25$, $r_\alpha = 0.3$, $\omega_h/\omega_\alpha = 0.2$, $a_h = -0.5$, $\mu = 100$ and no structural damping was included. These parameters were used by Crouse and Leishman[19] in the validation of their state space model.

Initially, a linear spring stiffness was assumed in order to compare the results from the analysis in this study and that of Ref.19 . Good agreement was obtained. The cubic nonlinearity $\bar{K}(\alpha)$ is given by equation (10), where $\beta_0 = 0$, $\beta_1 = 0.1$, $\beta_2 = 0$ and $\beta_3 =$ 40 which corresponds to parameters for a hardening spring and a pitch amplitude of 7 degrees. Figure 4 is a plot of the damping ratio with nondimensional speed for the flutter mode which is the bending mode in this case.

The flutter analysis was carried out using the describing function method. For these conditions, the nonlinearity decreases the flutter speed from 4.4 to approximately 3 which is a significant decrease. This flutter speed is a function of the Mach number assumed which is 0.85 for this case. Since the nonlinear spring stiffness is a function of the pitch amplitude, the variation of flutter speed with pitch amplitude is given in Figure 5 for two values of freestream Mach number. The results shown here agree with those obtained assuming incompressible flow in Reference 15 Indicating that there is a weak dependence on Mach number for this bifurcation plot. For each value of velocity there are two possible values of pitch amplitude indicating the possibility of a limit cycle oscillation. The maximum pitch amplitude was kept to less than ten degrees as it is believed that at higher values, the aerodynamic theory used here would not be accurate enough. Results obtained by Price et al would also experience the same restriction and in addition the assumption of incompressible flow would force the range of validity to lower amplitudes of pitch.

An interesting result from this comparison is that both results predict a supercritical Hopf bifurcation at $U/U^* \approx 0.22$. It is important to note that for a thin airfoil, the flowfield is subcritical at Mach number 0.7 but may become supercritical (for a NACA 64A006) at a Mach number of 0.85. However, in this case, the effect of compressibility is not significant.

For another set of parameters also used in Reference 15, $\beta_0 = 0$, $\beta_1 = 0.01$, $\beta_2 = 0$ and $\beta_3 = 5$, a linear flutter analysis was carried out by using the describing function method. For these conditions, the flutter velocity is much lower ($U_f = 0.68$) as shown in Figure 6 which is a plot of the damping ratio with the nondimensional flutter speed.

The results for this set of parameters in terms of U/U^* variation with pitch amplitude are shown in Figure 7. The decrease of flutter speed for the range of pitch amplitude from approximately 0 degrees to 6 degrees should be noted. This behaviour was not predicted in the work of Price et al who obtained a jump in flutter speed between 0 and 5 degrees denoting a discontinuity but did not find chaos in that region from a Lyaponov exponent analysis. Since the value of β_3 is lower for this example, it indicates a lower effective stiffness and a 'less hardening' spring.

A time response analysis with a more accurate numerical aerodynamic model, would determine the exact nature of the system in the range of amplitudes from 0 to 5 degrees. This can be compared with the results from the effective stiffness.

4. Conclusions

The aeroelastic stability of an airfoil with a structural nonlinearity in torsion was analyzed. A cubic nonlinearity was introduced to model the torsional stiffness. A state-space model was used for the aerodynamic loads and moments, enabling hard coupling of the fluid and structure for aeroelastic analysis. The introduction of the nonlinearity dramatically reduced the flutter speeds and this flutter speed decreased with decreasing spring restoring moments. Alternatively, a time response analysis should be carried out with a more exact aerodynamic model to confirm the response of the system.

For the examples chosen here, good agreement was obtained in comparison with a similar study[15]. The effective use of the state space model is to obtain a fast estimation of approximate flutter trends within the range of validity of the aerodynamic model. In this case, the amplitude of pitch should be low enough to be within the valid range of the approximate aerodynamic theory.

The state space formulation appears to be useful for aeroelastic analysis of helicopter rotors. It is a cost effective method for carrying out repetitive aeroelastic stability analysis. As indicated by other authors, the incorporation of a cubic spring stiffness can introduce limit cycle oscillations prior to the flutter point. It is imperative to extend the results of that investigation to the transonic flow regime in which flutter instability is a critical issue. Although results for incompressible flow indicated instabilities at speeds lower than the linear flutter velocity, it is possible that in transonic flow the aerodynamic (out-of-phase) components in the form of aerodynamic damping, can serve as a stabilizing influence under certain conditions. In order to verify that a limit cycle oscillation or chaotic oscillations can occur, a time response computation of the system must be carried out. This analysis would determine at what percentage of the flutter velocity, limit cycle or chaotic oscillations would occur for the transonic flows considered in this investigation.

References

1 . Woolston, D. S., H. L. Runyan and R. E. Andrews, "An Investigation of Effects of Certain Types of Structural Nonlinearities on Wing and Control Surface Flutter", Journal of the Aeronautical Sciences, 24, 1957, 57-63.

2 . Shen, S. F., "An Approximate Analysis of Nonlinear Flutter Problems", Journal of the Aerospace Sciences, 28, 1959, 25-32.

3 . Bogoliubov, N. N. And Mitropolsky, Y. A., "Asymptotic Methods in the Theory of Non-liner Oscillations, Hindustan, Delhi, 1985.

4 . Laurenson, R. M. and Trn, R. M.,"Flutter Analysis of Missile Control Surfaces Containing Structural Nonlinearities", AIAA Journal, 18, 1245-1251, 1980.

5 . McIntosh, S. C., Reed, R. E. And Rodden, W. P.,"Experimental and Theoretical Study of Nonlinear Flutter", Journal of Aircraft, 18, 1981, 1057-1063.

6 . Lee, C. L.,"An Iterative Procedure for Nonlinear Flutter Analysis", AIAA Journal, 24, 833-840, 1986.

7 . Ueda, T. And Dowell, E. H., "Flutter Analysis Using Nonlinear Aerodynamic Forces", Proceedings of the 23rd Structures, Structural Dynamics and Materials Conference, New Orleans, Louisiana, May 12-12, 1982.

8 . Dowell, E. H. And Ilgamov, M.,"Studies in Nonlinear Aeroelasticity", Springer-Verlag, 1988.

9 . Yang, Z. C. And I. C. Zhao, "Analysis of Limit cycle Flutter of an Airfoil in Incompressible Flow", Journal of Sound and Vibration, 123, 1988, 1-13.

10 . Lee, B. H. K. and Tron, A., "Effect of Structural Nonlinearities on Flutter Characteristics of the CF-18 Aircraft", Journal of Aircraft, 26, 1989, 781-786.

11 . Tjatra, I. W., Kapania, R. K. and Grossman, B."Transonic Flutter Analysis of Aerodynamic Surfaces in the Presence of Structural Nonlinearities", Computing Systems in Engineering, Vol. 1, Nos. 2-4, pp. 211-218, 1990.

12 . Hauenstein, A. Zara, J. Eversman, W., and Qumei, I.,"Chaotic and Nonlinear Dynamic Response of Aerosurfaces with Structural Nonlinearities", Proceedings of the 33rd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference, Part 4 Structural Dynamics II, Dallas, pp. 2367-2375.

13 . Tang, D. M. And Dowell, E. H., "Flutter and Stall Response of a Helicopter Blade with Structural Nonlinearity", Journal of Aircraft, Vol. 29, p. 953-960.

14 . Tanrikulu, O, Kuran, B., Ozguven, H. N. And Imregun, M., "Forced Harmonic Response Analysis of Nonlinear Using Describing Functions",AIAA Journal, Vol. 31, No. 7, July 1993.

15 . Price, S., Alighanbari, H. And Lee, B."The Aeroelastic Response of a Two-Dimensional Airfoil with Bilinear and Cubic Structural Nonlinearities, Proceedings of the AIAA/ASME/ASCE/AHS/ASC 35th Structures, Structural Dynamics and Materials Conference, April 18-20, 1994, Hilton Head, SC.

16 . Leishman, J. G. and Nguyen, K. Q., "State-Space Representation of Unsteady Airfoil Behaviour", AIAA Journal, Vol. 28, May 1990, pp. 836-844.

17 . Brietbach, E., "Effect of Structural Nonlinearities on Aircraft Vibration and Flutter", Presented at the 45th Structures and Materials AGARD Panel Meeting, Voss, Norway, AGARD Report 665, 1977

18 . Alighanbari, H. and Price, S. J., "Flutter Response of a Two-Dimensional Airfoil with a Hysteretic Structural Nonlinearity", Proceedings of the 12th Symposium on Engineering Applications of Mechanics, Interactions of Fluids, Structures and Mechanisms, June 27-29, 1994, McGill University, Montreal, Canada.

19 . Crouse, G. L. Jr. and Leishman, J. G., "Transonic Aeroelasticity Analysis Using State-Space Unsteady Aerodynamic Modeling", Journal of Aircraft, Vol. 29, Jan.-Feb. 1992, pp. 153-160.

20 . Edwards, J. W., Ashley, H., and Breakwell, J. V., "Unsteady Aerodynamic Modeling for Arbitrary Motions", AIAA Journal, Vol. 17, No. 4, 1979, pp. 365-374.

21. Leishman, J. G., "Validation of Approximate Indicial Aerodynamic Functions for Two-Dimensional Subsonic Flow", J. Aircraft, Vol. 25, No. 10, October 1988, pp. 914-922.

22 . Murty H. S., "Nonlinear Aspects of Transonic Aeroelasticity", Ph.D. dissertation, Institute for Aerospace Studies, University of Toronto, Toronto, 1992.(also University of Toronto Institute for Aerospace Studies Report No. 345).

23 . Murty, H. S. and Johnston, G. W.,"Nonlinear Aspects of Transonic Aeroelasticity", Canadian Aeronautics and Space Journal, Volume 39, No. 2, June 1993, pp. 78-84.

24 . Leishman, J. G. And Crouse, G. L, "A State-Space Model of Unsteady Aerodynamics for Flutter Analysis in a Compressible Flow", AIAA Paper 89-0022, Jan. 1989.

25 . Bathe, K. J. And Wilson, E. L., "Numerical Methods in Finite Element Analysis", Prentice-Hall, Englewood Cliffs, NJ, 1976.

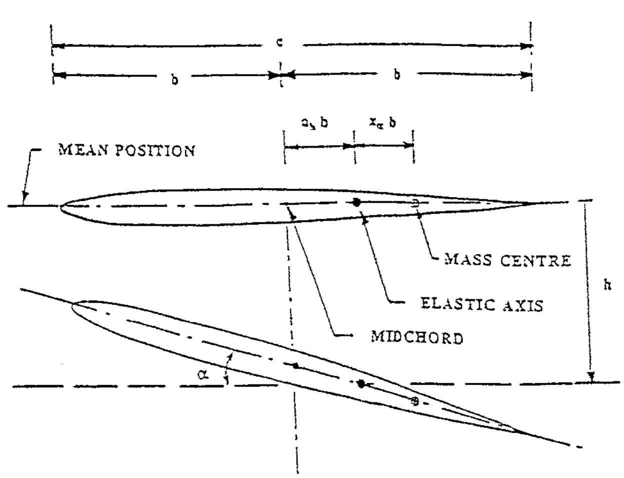

Figure 1.
Schematic representation of airfoil with motion in two degrees of freedom.

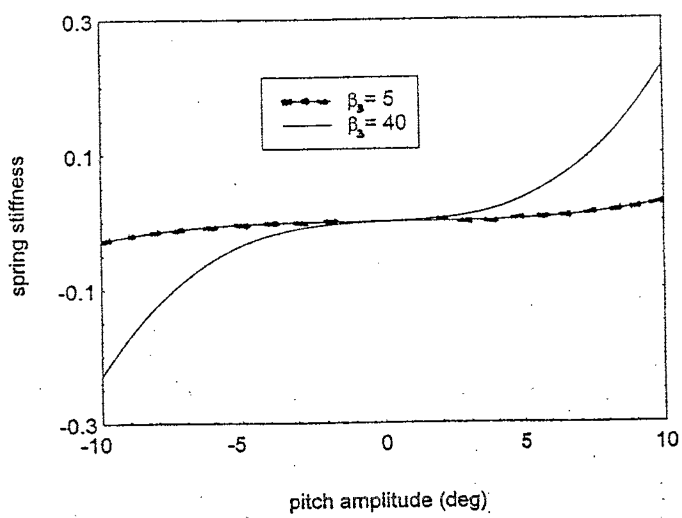

Figure 2 Variation of spring stiffness with pitch amplitude for varying hardening springs

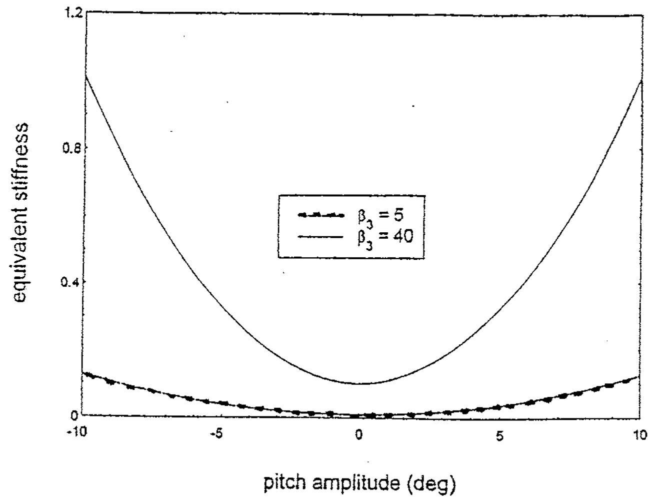

Figure 3 Effective stiffness variation with pitch amplitude for varying hardening springs

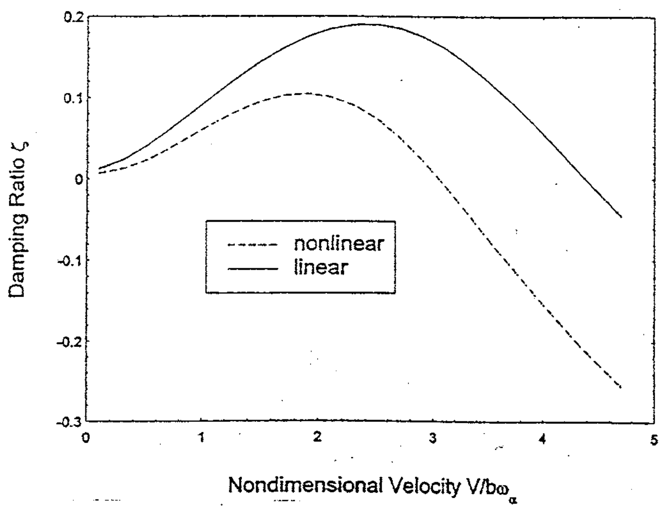

Figure 4 Damping ratio variation with nondimensional velocity for flutter calculation with β = 40

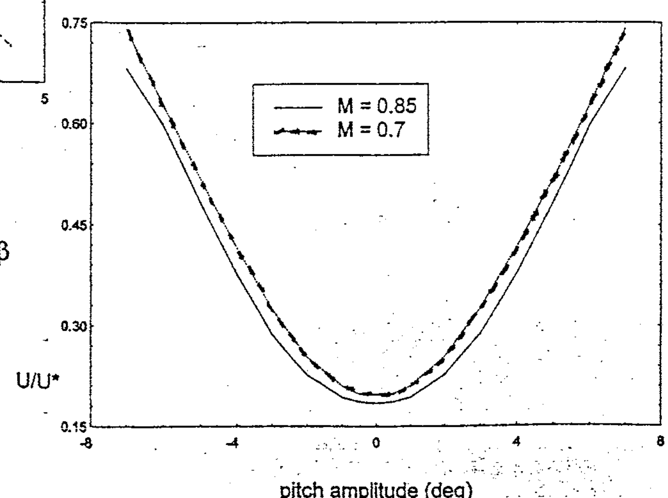

Figure 5 Ratio of nonlinear flutter speed to linear flutter speed with pitch amplitude for hardening spring β = 40

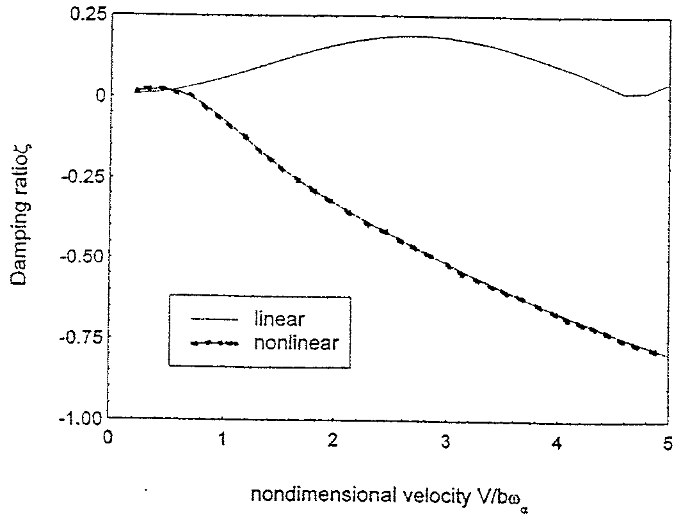

Figure 6 Damping ratio variation with nondimensional velocity for flutter calculation with β = 5

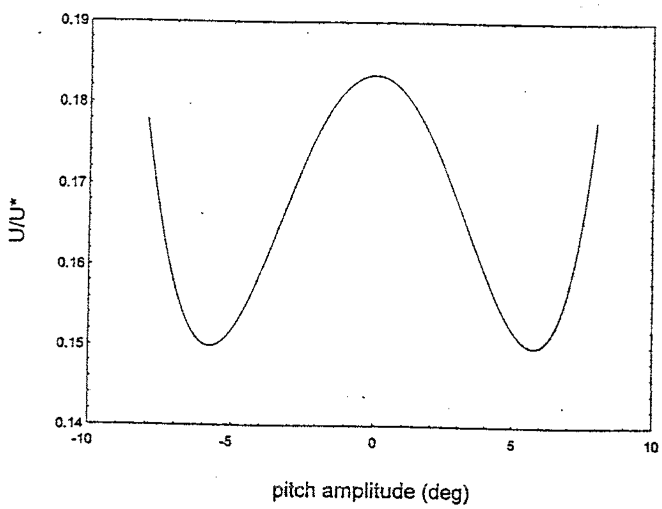

Figure 7 Ratio of nonlinear flutter speed to linear flutter speed with pitch amplitude for hardening spring β = 5