A POSTERIORI FINITE ELEMENT ERROR ESTIMATION FOR DIFFUSION PROBLEMS

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Abstract. We consider a posteriori estimates of spatial discretization errors of p^{th} order finite element solutions of two-dimensional elliptic and parabolic problems on meshes of rectangular elements. We show that error estimates for piecewise bi-p polynomial spaces obtained from jumps in solution gradients at element vertices when p is odd and from local elliptic or parabolic problems when p is even extend to other solution spaces. In particular, we establish that these error estimates converge at the same rate as the actual error for finite element spaces that contain all two-dimensional monomial terms of order $p+1$ except for x_1^{p+1} and x_2^{p+1} in a Cartesian frame with coordinates (x_1, x_2). Computational results show that the error estimates are accurate and robust for a wide range of problems, including some that are not supported by the present theory. These involve quadrilateral-element meshes, singularities, and nonlinearity.

Key Words. Finite element methods, a posteriori error estimation, p-refinement, hierarchical approximations, elliptic and parabolic partial differential equations.

AMS (MOS) subject classifications. 65M60, 65M20, 65M15, 65M50.

1. Introduction. A posteriori error estimates are a standard ingredient of adaptive finite element software. They are used to appraise the accuracy of computed solutions and to control adaptive enrichment through $h-$, $p-$, and/or r-refinement. Successful techniques for estimating spatial discretization errors of finite element solutions of elliptic and

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parabolic problems are often based on residual correction with p-refinement [13]. Using this strategy, an error estimate is obtained in a space of piecewise polynomials having higher degree than used for the original solution by solving a finite element Galerkin problem with solution residuals as loading. The error estimation problem may be localized to the element level to avoid a global assembly and solution; hence, reducing computational cost. Localization typically involves estimating solution gradients at element boundaries [4, 6, 13] and the use of ''superconvergence'' properties [3, 5, 14, 15] to neglect errors at certain points, lines, or surfaces.

Babuska and Yu [5] considered the solution of linear two-dimensional elliptic problems on squares using piecewise bi-p polynomial spaces and showed that error estimates could be constructed from jumps in solution gradients at element vertices when p is odd and from local elemental solution residuals when p is even. Yu [14, 15] proved that error estimates computed in this manner are asymptotically exact; hence, they converge to zero under mesh refinement at the same rate as the actual finite element error. Adjerid et al. [2, 3] established similar results for the finite element method-of-lines solution of one- and two-dimensional parabolic problems. These error estimates are efficient and suitable for parallel computation, with the even-degree estimates requiring no off-element references.

Piecewise tensor-product spaces are not as efficient as serendipity [16] or hierarchical [8, 12] approximations which have fewer degrees of freedom for the same order of accuracy. Herein, we show that the error estimates of Babuska and Yu [5, 14, 15] or Adjerid et al. [2, 3] converge to the actual error for a wider class of finite element approximations. The important consideration is that a solution space of order p contain all monomial terms $x_1^{p+1-k}x_2^k$ of degree $p+1$ except x_1^{p+1} and x_2^{p+1}. These spaces have slightly larger dimension than the usual serendipity or hierarchical bases, but far less than the bi-p spaces (cf. Fig. 1). In return, for the modest increase in solution complexity relative to serendipity and hierarchical bases, the solution will be supported by a simple and asymptotically correct error estimate.

After stating the linear elliptic and parabolic Galerkin problems under consideration (§2), we review the error estimation procedures of Babuska and Yu [5, 14, 15] and Adjerid et al. [2, 3] for piecewise bi-p polynomial spaces (§3). The procedure for constructing more general finite element spaces for which the aforementioned error estimates apply is described in §4. Establishing asymptotic correctness of the error estimates for elliptic and parabolic problems using these new finite element spaces follows the earlier arguments used for bi-p polynomial spaces. Because these proofs are lengthy and involved, we have not duplicated the arguments but, rather, refer to the earlier analyses [2, 3, 5, 14, 15].

Although many finite element spaces satisfying the theory can be constructed, we show how to modify standard hierarchical spaces [12] by adding certain interior ("bubble") modes to them (§4). The error estimates of Babuska and Yu [5, 14, 15] or Adjerid et al. [2, 3] may be used with these spaces to appraise the accuracy of finite element solutions of linear elliptic and parabolic problems, respectively, on meshes of rectangular elements. However, using several examples, we show (§5) that the error estimates work more generally than the present theory would suggest. In particular, they appear be reliable and robust on some non-rectangular-element meshes, highly-graded meshes in the presence of singularities, and nonlinear problems. Unfortunately, computational evidence also indicates a performance degradation of the error estimates when elements are severely distorted from rectangular or when meshes lack proper grading near singularities. Similar effects were noted by Babuska and Yu [5]. Thus, additional theory and modification may be necessary (§6).

2. Problem formulation. Consider the linear, scalar, two-dimensional parabolic partial differential equation

$$\partial_t + \mathbf{L}u = f(\mathbf{x}), \quad \mathbf{x} = [x_1, x_2]^T \in \Omega, \tag{2.1a}$$

with

$$\mathbf{L}u = -\sum_{j=1}^{2}\sum_{k=1}^{2}\partial_{x_j}(a_{j,k}(\mathbf{x})\partial_{x_k}u) + b(\mathbf{x})u, \tag{2.1b}$$

subject to the initial and Dirichlet boundary conditions

$$u(\mathbf{x},0) = u^0(\mathbf{x}), \quad \mathbf{x} \in \Omega \cup \partial\Omega, \tag{2.1c}$$

$$u(\mathbf{x},t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad t > 0. \tag{2.1d}$$

The variables $\mathbf{x} = [x_1, x_2]^T$ and t denote spatial and temporal coordinates, ∂_α denotes partial differentiation with respect to α, and Ω is a bounded piecewise rectangular domain with boundary $\partial\Omega$. The functions $a_{j,k}(\mathbf{x})$, $j, k = 1, 2$, and $b(\mathbf{x})$ are smooth with $\mathbf{L}$ being a positive definite operator. Our results also hold for elliptic problems upon neglect of temporal dependence in (2.1).

The Galerkin form of (2.1) consists of determining $u \in H_0^1$ satisfying

$$(v, \partial_t u) + A(v,u) = (v,f), \quad t > 0, \tag{2.2a}$$

$$A(v,u) = A(v,u^0), \quad t = 0, \quad \text{for } all \ v \in H_0^1, \tag{2.2b}$$

where the strain energy and L^2 inner products, respectively, are

$$A(v,u) = \iint_\Omega [\sum_{j=1}^{2}\sum_{k=1}^{2} a_{j,k}(\mathbf{x})\partial_{x_j}v\,\partial_{x_k}u + b(\mathbf{x})vu]\,dx_1dx_2 \tag{2.2c}$$

and

$$(v,u) = (v,u)_0 = \iint_\Omega uv\ dx_1dx_2. \tag{2.2d}$$

As usual, functions in the Sobolev space H^s, $s \geq 0$, have the inner product and norm

$$(v,u)_s = \sum_{|\boldsymbol{\alpha}|\leq s} (\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}v, \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}u), \quad \|u\|_s^2 = (u,u)_s, \tag{2.2e,f}$$

where $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2$ with α_1 and α_2 being non-negative integers. A 0 subscript on H^1 implies that functions also satisfy (2.1d).

Finite element solutions of (2.2a,b) are obtained by approximating H^1 by a finite-dimensional subspace $S^{N,p}$ and determining $U \in S_0^{N,p}$ such that

$$(V,\partial_t U) + A(V,U) = (V,f), \quad t > 0, \tag{2.3a}$$

$$A(V,U) = A(V,u^0), \quad t = 0, \quad \text{for } all \ V \in S_0^{N,p}. \tag{2.3b}$$

Partitioning Ω into a mesh of rectangular elements Δ_i, $i = 1, 2, \cdots, N$, define $S^{N,p}$ as

$$S^{N,p} = \{\, w \in H^1 \mid w(\mathbf{x}) \in Q_p(\Delta_i), \quad \mathbf{x} \in \Delta_i, \quad i = 1, 2, \cdots, N \,\} \tag{2.4}$$

where Q_p involves polynomials of order $p \geq 1$ that will be defined in §3 and §4 for bi-p and hierarchical approximations, respectively.

3. Error estimation with piecewise bi-p polynomial approximation. The error estimates of Babuska and Yu [5, 14, 15] and Adjerid et al. [2] involve solution spaces $S_0^{N,p}$ on meshes of square elements with Q_p defined as a tensor product of one-dimensional polynomials through degree p. With

$$e(\mathbf{x},t) = u(\mathbf{x},t) - U(\mathbf{x},t) \tag{3.1}$$

denoting the discretization error of the semi-discrete problem (2.3), we summarize their results for obtaining H^1 error estimates $E(\cdot,t)$ for odd- and even-degree approximations (cf. §3.1 and §3.2, respectively).

To begin, define [3, 14] a univariate interpolation operator $\bar{\pi}$ that maps functions in $H_0^1[-1,1]$ onto the space of univariate polynomials of degree $p \geq 1$. Let $p = 2l - 1$ or $2l$ according to whether it is odd or even, respectively, and (i) place $2l$ interpolation points $\pm\xi_j$, $j = 1, 2, \cdots, l$, symmetrically disposed with respect to the origin; (ii) set $\xi_l = 1$; (iii) set $\xi_0 = 0$ when p is even; and (iv) determine the remaining point locations to satisfy

$$\int_{-1}^{1} \bar{\Psi}'_{p+1}(\xi)\xi^s \, d\xi = 0, \quad s = 0, 1, \cdots, p-1, \tag{3.2a}$$

where

$$\bar{\Psi}_{p+1}(\xi) = \xi^{p+1} - \bar{\pi}\xi^{p+1} \tag{3.2b}$$

and $(\,)'$ denotes total differentiation. These considerations imply that $\bar{\psi}'_{p+1}(\xi)$ is proportional to the Legendre polynomial $P_p(\xi)$ of degree p [3].

Continuing, let $\hat{\pi}$ denote the two-dimensional operator that interpolates functions in $H_0^1[-1,1]\times[-1,1]$ as tensor products of the one-dimensional interpolants $\bar{\pi}$ in the ξ_1 and ξ_2 directions. Interpolants π on Δ_i are defined as $\pi_i f(\mathbf{x}) = \hat{\pi} f(\mathbf{x}(\xi_1,\xi_2))$, where $\mathbf{x}(\xi_1,\xi_2)$ denotes the bilinear mapping from the "canonical element" $\{ (\xi_1,\xi_2) \mid -1 \le \xi_1,\ \xi_2 \le 1 \}$ to Δ_i. We omit the elemental index i whenever confusion is unlikely.

3.1. Error estimates for odd-degree approximations. Error estimates of odd-degree approximations are constructed by assuming

$$e(\mathbf{x},t) \approx E(\mathbf{x},t) = b_1(t)\psi_{p+1,1}(\mathbf{x}) + b_2(t)\psi_{p+2,2}(\mathbf{x}) \tag{3.3a}$$

with

$$\psi_{p+1,j}(\mathbf{x}) = x_j^{p+1} - \pi x_j^{p+1}, \quad j = 1, 2, \quad \mathbf{x} \in \Delta_i. \tag{3.3b,c}$$

Assuming that $u(\mathbf{x},t) \in C^1(\Omega)$, use (3.1) to compute jumps in the spatial derivatives of $e(\mathbf{x},t)$ at the vertices $\mathbf{p}_k$, $k = 1, 2, 3, 4$, of Δ_i as

$$[\partial_{x_j} e(\mathbf{p}_k,t)]_j = -[\partial_{x_j} U(\mathbf{p}_k,t)]_j = b_1(t)[\partial_{x_j}\psi_{p+1,1}(\mathbf{p}_k)]_j + b_2(t)[\partial_{x_j}\psi_{p+1,2}(\mathbf{p}_k)]_j,$$

$$j = 1, 2, \quad k = 1, 2, 3, 4, \quad \mathbf{x} \in \Delta_i, \tag{3.4}$$

where $[q(\mathbf{p})]_j$ denotes the jump in q at point $\mathbf{p}$ the x_j direction.

The jumps $[\partial_{x_j} U(\mathbf{p}_k,t)]_j$ in the finite element solution derivatives are known in each coordinate direction ($j = 1,2$) at each vertex ($k = 1, 2, 3, 4$) for elements not adjacent to $\partial\Omega$. Thus, four solutions $b_{1,k}$, $b_{2,k}$, $k = 1, 2, 3, 4$, can be obtained from (3.4) as

$$b_{1,k}(t)[\partial_{x_j}\psi_{p+1,1}(\mathbf{p}_k)]_j + b_{2,k}(t)[\partial_{x_j}\psi_{p+1,2}(\mathbf{p}_k)]_j = -[\partial_{x_j} U(\mathbf{p}_k,t)]_j$$

$$j = 1, 2, \quad k = 1, 2, 3, 4, \quad \mathbf{x} \in \Delta_i. \tag{3.5}$$

These, in turn, can be used with (3.3a) to compute four error estimates on Δ_i which are averaged to obtain

$$\|E(\cdot,t)\|_{1,i}^2 = \tfrac{1}{4}\sum_{k=1}^{4} \|b_{1,k}(t)\psi_{p+1,1}(\mathbf{x}) + b_{2,k}(t)\psi_{p+1,2}(\mathbf{x})\|_{1,i}^2, \quad \mathbf{x} \in \Delta_i, \tag{3.6a}$$

where the local H^1 norm $\| \|_{1,i}$ is defined like its global counterpart with Δ_i replacing Ω in (2.2). A global error estimate is obtained as

$$\|E(\cdot,t)\|_1^2 = \sum_{i=1}^{N} \|E(\cdot,t)\|_{1,i}^2 . \tag{3.6b}$$

When Δ_i is adjacent to $\partial\Omega$, (3.6a) is obtained either by averaging (3.5) over the interior vertices of Δ_i or by solving (3.5) in a least-squares sense using jumps at vertices across all edges of Δ_i except $\partial\Omega$.

Error estimates computed in this manner are asymptotically correct as indicated by the following theorem.

THEOREM 1. *Let Ω be a rectangle that has been partitioned into N rectangular $h_{i,1} \times h_{i,2}$ elements Δ_i, $i = 1, 2, \cdots, N$. Let positive constants c and C exist such that*

$$c \le \frac{h_{adj(i,k),j}}{h_{i,j}} \le C, \quad j = 1, 2, \quad k = 1, 2, 3, 4, \quad i = 1, 2, \cdots, N, \tag{3.7}$$

where $h_{adj(i,k),j}$ is the length of the edge of the element adjacent to Δ_i in the x_j direction and sharing vertex k. Further let $u \in H_0^1 \cap H^{p+2}$ and $U \in S_0^{N,p}$ be solutions of (2.2) and (2.3), respectively, where $p \ge 1$ is an odd integer. Then

$$\|e(\cdot,t)\|_1^2 = \|E(\cdot,t)\|_1^2 + O(h^{2p+1}) \tag{3.8a}$$

where $h = \max_{i=1,2,\cdots,N} \max_{j=1,2} (h_{i,j})$ and

$$\|E(\cdot,t)\|_1^2 = \frac{1}{4(2p+1)} \sum_{i=1}^{N} h_{i,1} h_{i,2} \sum_{j=1}^{2} \sum_{k=1}^{4} \left[\frac{[\partial_{x_j} U(\mathbf{p}_k,t)]_j}{1 + (h_{adj(i,k),j}/h_{i,j})^p} \right]^2 . \tag{3.8b}$$

Proof. Adjerid et al. [2] and Yu [15] established convergence, respectively, for parabolic and elliptic problems on square domains. The extension of their results to rectangular domains and elements is straight forward. □

3.2. Error estimates for even-degree approximations. When p is even, we construct a Galerkin problem for e by replacing u in (2.2a,b) by $U + e$ to obtain

$$(v, \partial_t e) + A(v, e) = g(t, v), \quad t > 0, \tag{3.9a}$$

$$A(v, e) = A(v, u^0 - U), \quad t = 0, \quad \text{for } all \ v \in H_0^1, \tag{3.9b}$$

with

$$g(t, v) = (v, f) - (v, \partial_t U) - A(v, U). \tag{3.9c}$$

The trial function (3.3) is again used to approximate e while the approximation of the test function v is selected as

$$V_j(\mathbf{x}) = \psi_{p+1,j}(\mathbf{x})\delta(\xi_{(j \bmod 2)+1}(\mathbf{x})), \quad j = 1, 2, \quad \mathbf{x} \in \Delta_i, \tag{3.10a}$$

where

$$\delta(\xi) = \frac{\overline{\psi}_{p+1}(\xi)}{\xi}, \quad \sigma(\xi) = \psi'_{p+1}(\xi), \quad \xi \in [-1,1], \tag{3.10b,c}$$

and $\xi_j(\mathbf{x})$, $j = 1, 2$, is a bilinear mapping of Δ_i to $[-1,1] \times [-1,1]$.

The functions $\psi_{p+1,j}(\mathbf{x})$ and $V_j(\mathbf{x})$, $j = 1, 2$, vanish on the edges of Δ_i; thus, the computation of $E(\mathbf{x}, t)$ is local to Δ_i and is obtained as the solution of

$$(V_j, \partial_t E)_i + A_i(V_j, E) = g_i(t, V_j), \quad t > 0, \tag{3.11a}$$

$$A_i(V_j, E) = A_i(V_j, u^0 - U), \quad t = 0, \quad j = 1, 2, \tag{3.11b}$$

where the subscript i denotes a local inner product whose domain is restricted to Δ_i. This problem may be further simplified by (i) neglecting the off-diagonal diffusion coefficients $a_{j,k}$, $j \neq k$, and the reaction term $b(\mathbf{x})$ as being higher-order; (ii) approximating the diagonal diffusion coefficients $a_{j,j}$, $j = 1, 2$, by their values at element centroids; and (iii) using the symmetry of $\psi_{p+1,j}(\mathbf{x})$ and $V_j(\mathbf{x})$, $j = 1, 2$, to obtain the uncoupled constant-coefficient initial-value problem on Δ_i

$$b'_j(t) + r_j b_j(t) = G_j(t), \quad t > 0, \tag{3.12a}$$

$$b_j(0) = \frac{(h_{i,j}/2)^{-2p+3}}{\overline{a}_{j,j}(h_{i,(j \bmod 2)+1}/2)} \frac{A_i(V_j, u^0(\cdot) - U(\cdot,0))}{\int_{-1}^{1}\int_{-1}^{1} \sigma^2(\xi_j)\delta(\xi_{(j \bmod 2)+1})\, d\xi_1 d\xi_2}, \quad j = 1, 2, \tag{3.12b}$$

where

$$r_j = \frac{\bar{a}_{j,j}}{(h_{i,j}/2)^2} \frac{\int_{-1}^{1}\int_{-1}^{1} \sigma^2(\xi_j)\delta(\xi_{(j \bmod 2)+1})\, d\xi_1 d\xi_2}{\int_{-1}^{1}\int_{-1}^{1} \psi_{p+1}^2(\xi_j)\delta(\xi_{(j \bmod 2)+1})\, d\xi_1 d\xi_2}, \tag{3.12c}$$

$$G_j(t) = \frac{(h_{i,j}/2)^{-(2p+3)}}{(h_{i,(j \bmod 2)+1}/2)} \frac{g_i(t, V_j)}{\int_{-1}^{1}\int_{-1}^{1} \psi_{p+1}^2(\xi_j)\delta(\xi_{(j \bmod 2)+1})\, d\xi_1 d\xi_2}, \tag{3.12d}$$

and $\bar{a}_{j,j}$ denotes the value of $a_{j,j}$, $j = 1, 2$, at the centroid of Δ_i.

Further simplification is afforded by neglecting the time derivative in (3.12a) to obtain

$$b_j(t) = \frac{(h_{i,j}/2)^{-2p+1}}{\bar{a}_{j,j}(h_{i,(j \bmod 2)+1}/2)} \frac{g_i(t, V_j)}{\int_{-1}^{1}\int_{-1}^{1} \sigma^2(\xi_j)\delta(\xi_{(j \bmod 2)+1})\, d\xi_1 d\xi_2}, \quad t > 0, \quad j = 1, 2. \tag{3.13}$$

Thus, error estimates may be determined as solutions of either local parabolic (3.12) or elliptic (3.13) problems. Both methods produce asymptotically correct results as indicated by the following theorem.

THEOREM 2. *Let the mesh and solution structure be as described in Theorem 1 and let* $p \geq 2$ *be an even integer. Let* b_j, $j = 1, 2$, *be solutions of either (3.12) or (3.13) that are used to obtain an error estimate according to (3.3). Then, there exists a constant* $\delta > 0$ *such that*

$$\|e(\cdot,t)\|_1^2 = \|E(\cdot,t)\|_1^2 + O(h^{2p+1}), \quad t > \delta, \tag{3.14a}$$

where

$$\|E(\cdot,t)\|_1^2 = \sum_{i=1}^{N}\sum_{j=1}^{2} b_{j,i}^2(t)(h_{i,j}/2)^{2p+1}(h_{i,(j \bmod 2)+1}/2)\int_{-1}^{1}\int_{-1}^{1} \sigma^2(\xi_j)\, d\xi_1 d\xi_2. \tag{3.14b}$$

Proof. cf. Adjerid et al. [2, 3] for parabolic problems and Yu [14] for elliptic problems on square domains. Again, the extension to rectangular meshes is straight forward. □

4. Error estimation for other finite element bases. With sufficient smoothness, the convergence rate of finite element solutions is determined by the highest degree polynomial that can be interpolated exactly. Thus, piecewise bi-p polynomial approximations contain many higher-order terms that do not increase the convergence rate. Different bases of order p, such as serendipity [16] or hierarchical [8, 12] approximations, typically lead to (i) better conditioned stiffness matrices, (ii) reduced computational complexity, and (iii) simpler implementations. Unfortunately, the error estimation procedures (3.6, 8, 12-14) are not asymptotically correct when used with these spaces.

The terms x_1^{p+1} and x_2^{p+1} are the only monomials missing from a bi-p polynomial approximation for it to contain a complete $(p+1)$-degree polynomial (cf. Fig. 1). As indicated by the following theorem, these are the only monomial terms needed to make the error estimates of §3 asymptotically correct when the other monomial terms of degree $p+1$ are present in the solution space $S^{N,p}$.

THEOREM 3. *Under the conditions of Theorems 1 and 2, let $Q_p(\Delta_i)$ be the restriction of $S^{N,p}$ to Δ_i (cf. (2.4)) and let $M_p(\Delta_i)$ be a space of complete polynomials of degree p on Δ_i. If Q_p satisfies*

$$M_p \subset Q_p \subset M_{p+1}, \quad M_{p+1} \subset Q_p \cup \{ x_1^{p+1}, x_2^{p+1} \} \tag{4.1}$$

then the error estimates (3.8) or (3.14) apply when p is odd or even, respectively.

Proof. Again, the proof closely parallels those of Adjerid et al. [2, 3] and Yu [14, 15]. □

Conditions (4.1) may be used to construct many solution spaces where the error estimates of §3 are asymptotically correct; however, let us focus on the hierarchical basis of

Szabo and Babuska [12]. Letting

$$\bar{\phi}^1_{\pm 1}(\xi) = \tfrac{1}{2}(1 \pm \xi), \quad \bar{\phi}^k_0(\xi) = \sqrt{\frac{2k-1}{2}} \int_{-1}^{\xi} P_{k-1}(\zeta)\, d\zeta, \quad k = 2, 3, \cdots, p, \tag{4.2}$$

be the one-dimensional basis, then the two-dimensional hierarchical basis with respect to the canonical square element contains

i. the four vertex shape functions

$$N_1(\xi_1,\xi_2) = \bar{\phi}^1_{-1}(\xi_1)\bar{\phi}^1_{-1}(\xi_2), \quad N_2(\xi_1,\xi_2) = \bar{\phi}^1_1(\xi_1)\bar{\phi}^1_{-1}(\xi_2), \tag{4.3a,b}$$

$$N_3(\xi_1,\xi_2) = \bar{\phi}^1_1(\xi_1)\bar{\phi}^1_1(\xi_2), \quad N_4(\xi_1,\xi_2) = \bar{\phi}^1_{-1}(\xi_1)\bar{\phi}^1_1(\xi_2); \tag{4.3c,d}$$

ii. the $4(p-1)$ edge shape functions

$$N^k_1(\xi_1,\xi_2) = \bar{\phi}^1_{-1}(\xi_2)\bar{\phi}^k_0(\xi_1), \quad N^k_2(\xi_1,\xi_2) = \bar{\phi}^1_1(\xi_1)\bar{\phi}^k_0(\xi_2), \tag{4.4a,b}$$

$$N^k_3(\xi_1,\xi_2) = \bar{\phi}^1_1(\xi_1)\bar{\phi}^k_0(\xi_1), \quad N^k_4(\xi_1,\xi_2) = \bar{\phi}^1_{-1}(\xi_1)\bar{\phi}^k_0(\xi_2), \quad k = 2, 3, \cdots, p; \tag{4.4c,d}$$

iii. the $(p-2)(p-3)/2$ internal shape functions

$$N^1_0(\xi_1,\xi_2) = 4\bar{\phi}^1_{-1}(\xi_1)\bar{\phi}^1_1(\xi_1)\bar{\phi}^1_{-1}(\xi_2)\bar{\phi}^1_1(\xi_2), \tag{4.5a}$$

$$N^2_0(\xi_1,\xi_2) = N^1_0(\xi_1,\xi_2)P_1(\xi_1), \quad N^3_0(\xi_1,\xi_2) = N^1_0(\xi_1,\xi_2)P_1(\xi_2), \tag{4.5b,c}$$

$$N^4_0(\xi_1,\xi_2) = N^1_0(\xi_1,\xi_2)P_2(\xi_1), \quad N^5_0(\xi_1,\xi_2) = N^1_0(\xi_1,\xi_2)P_1(\xi_1)P_1(\xi_2), \tag{4.5d,e}$$

$$N^6_0(\xi_1,\xi_2) = N^1_0(\xi_1,\xi_2)P_2(\xi_2), \quad \cdots, \quad N^{(p-2)(p-3)/2}_0(\xi_1,\xi_2) = N^1_0(\xi_1,\xi_2)P_{p-4}(\xi_2). \tag{4.5f,g}$$

The four vertex shape functions (4.3) are the usual bilinear shape functions that vanish on the two edges opposite the vertex to which they are associated. Edge functions (4.4) are present in the basis for $p \geq 2$. They are nonzero only on one element edge and decrease linearly in a direction normal to this edge. The interior "bubble" functions (4.5) are present when $p \geq 4$ and vanish on all element edges. The monomial terms that are present in this hierarchical approximation when $p = 4$ are shown in Fig. 1.

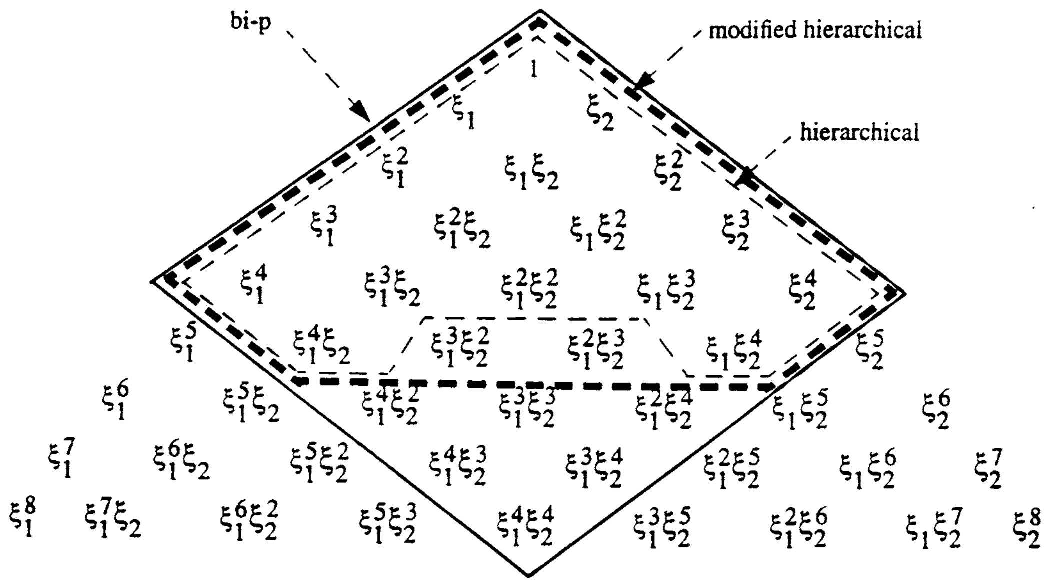

FIG. 1. *Pascal triangle showing the monomial terms present in a bi-p polynomial, a p-degree hierarchical polynomial, and a modified p-degree hierarchical polynomial satisfying conditions (4.1) for* $p = 4$.

The hierarchical basis (4.3-5) does not satisfy conditions (4.1) for $p \geq 3$; however, the space Q_p obtained by adding the interior shape functions associated with the hierarchical basis of order $p + 1$ and the hierarchical basis of order p does satisfy (4.1). As shown in Fig. 1, the only terms missing from this modified hierarchical space are ξ_1^{p+1} and ξ_2^{p+1}. As an example, we list the minimal sets Q_p, $p = 1, 2, 3, 4$, that satisfy (4.1) in terms of the hierarchical basis (4.3-5)

$$Q_1 = \{ N_1, N_2, N_3, N_4 \}, \tag{4.6a}$$

$$Q_2 = Q_1 \cup \{ N_1^2, N_2^2, N_3^2, N_4^2 \}, \tag{4.6b}$$

$$Q_3 = Q_2 \cup \{N_1^3, N_2^3, N_3^3, N_4^3\} \cup \{N_0^1\}, \tag{4.6c}$$

$$Q_4 = Q_3 \cup \{N_1^4, N_2^4, N_3^4, N_4^4\} \cup \{N_0^2, N_0^3\}. \tag{4.6d}$$

As noted, the sets Q_1 and Q_2 are identical to the usual hierarchical basis (4.3,4). The set Q_3 differs from the usual hierarchical basis by the bubble function N_0^1, which is normally associated with the hierarchical basis of degree four. Likewise Q_4 contains the internal modes N_0^2 and N_0^3, which are usually associated with a fifth-degree basis. The addition of these internal modes in a finite element software system is simple and results in a minor loss in efficiency relative to the standard hierarchical basis. In return for this extra effort, the solution will be supported by a simple asymptotically correct error estimate.

5. Examples. We consider five examples that illustrate the performance of the error estimation procedures for both odd- and even-degree approximations by solving elliptic and parabolic problems having (i) smooth solutions, (ii) solutions with line and point singularities, (iii) nonuniform and highly-graded quadrilateral meshes, and (iv) nonlinearity. The assumptions of Theorems 1-3 are violated for all examples; thus, indicating that the estimation procedures apply more widely than the theory suggests.

Accuracy of the error estimates is measured by the global and local *effectivity indices*

$$\theta = \frac{\|E(\cdot,t)\|_1}{\|e(\cdot,t)\|_1}, \quad \theta_i = \frac{\|E(\cdot,t)\|_{1,i}}{\|e(\cdot,t)\|_{1,i}}, \quad i = 1, 2, \cdots, N. \tag{5.1}$$

If E is an asymptotically correct estimate of e then θ should converge to unity as the mesh is refined. The estimate is, furthermore, robust if θ does not appreciably differ from unity for a wide range of mesh spacings and polynomial degrees.

Example 1. Consider Poisson's equation

$$\Delta u = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{5.2a}$$

on the quadrilateral domain Ω with vertices at (0.5,-0.5), (2.0,0.4), (2.5,1.2), and (0.0,2.0). Let $f(\mathbf{x})$ and Dirichlet boundary conditions be selected such that the exact solution is

$$u(x_1,x_2) = e^{-2x_1x_2}. \tag{5.2b}$$

We solve (5.2) on uniform quadrilateral-element meshes having 100, 225, 400, 625, and 900 elements using bi-p, modified hierarchical (cf. (4.6)), and standard hierarchical (cf. (4.3-5) piecewise polynomial approximations of orders 1 to 4. Errors and global effectivity indices in the H^1 norm are presented in Tables 1-3 for piecewise bi-p, modified hierarchical, and standard hierarchical approximations, respectively. Results for bi-p and modified hierarchical approximations are good with effectivity indices in excess of 0.85 for virtually all mesh-order combinations. Results for this problem, which has a smooth solution and reasonable quadrilateral meshes, also indicate that the effectivity indices of both approximations are converging to unity under mesh refinement. On the contrary, the results of Table 3 for the standard hierarchical basis do not indicate asymptotic correctness of the error estimates for $p = 3, 4$. Recall (cf. §4.) that the modified and standard hierarchical bases agree when $p < 3$.

We also solve this problem on a chevron-patterned mesh obtained by mapping the vertices

$$\zeta_{1,j} = \frac{j}{n}, \quad \zeta_{2,k} = \frac{k}{n} + \frac{(-1)^{j+k}}{3n}\begin{cases} 1, & \text{if } k \in [1, n-1] \\ 0, & \text{if } k = 0, n \end{cases}, \quad j, k = 1, 2, \cdots, n = \sqrt{N}, \tag{5.3}$$

of $[0,1]\times[0,1]$ onto corresponding vertices in Ω by a bilinear transformation and forming quadrilateral elements. We solve problems with $p = 1, 2, 3, 4$ and $N = 100, 225, 400, 625$ using bi-p approximations. Errors and global effectivity indices are displayed in Table 4. Results for piecewise modified hierarchical approximations are similar. While effectivity indices are in excess of 0.8, convergence under mesh refinement on these quadrilateral meshes is less clear than on uniform quadrilateral-element meshes.

Example 2. Consider Poisson's equation (5.2a) on a unit square with $f(\mathbf{x})$, the Neumann boundary conditions on $x_2 = 0, 1$, and the Dirichlet boundary conditions on $x_1 = 0, 1$ specified so that the exact solution is

TABLE 1
Errors and effectivity indices for Example 1 using piecewise bi-p polynomial approximations.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
100	0.294	0.823	0.283(−1)	0.972	0.231(−2)	0.653	0.159(−3)	0.941
225	0.199	0.890	0.130(−1)	0.987	0.716(−3)	0.777	0.332(−4)	0.972
400	0.150	0.926	0.738(−2)	0.992	0.307(−3)	0.845	0.107(−4)	0.984
625	0.120	0.947	0.475(−2)	0.995	0.158(−3)	0.887	0.444(−5)	0.990
900	0.100	0.960	0.331(−2)	0.996	0.922(−3)	0.914	0.215(−5)	0.993

TABLE 2
Errors and effectivity indices for Example 1 using piecewise modified hierarchical polynomial approximations.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
100	0.294	0.823	0.284(−1)	0.966	0.237(−2)	0.483	0.176(−3)	0.850
225	0.199	0.890	0.130(−1)	0.983	0.725(−3)	0.673	0.350(−4)	0.927
400	0.150	0.926	0.739(−2)	0.990	0.310(−3)	0.779	0.110(−4)	0.959
625	0.120	0.947	0.475(−2)	0.994	0.159(−3)	0.841	0.452(−5)	0.974
900	0.100	0.960	0.331(−2)	0.996	0.925(−3)	0.881	0.218(−5)	0.982

TABLE 3
Errors and effectivity indices for Example 1 using piecewise hierarchical polynomial approximations.

p	3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ
100	0.463(−2)	0.903	0.693(−3)	0.289
225	0.140(−2)	1.090	0.143(−3)	0.303
400	0.580(−3)	1.208	0.452(−4)	0.313
625	0.291(−3)	1.288	0.183(−4)	0.321
900	0.165(−3)	1.344	0.876(−5)	0.327

$$u(x_1,x_2) = x_2^{3/2}. \tag{5.4}$$

This one-dimensional solution has a line singularity at $x_2 = 0$. Numerical results using piecewise bi-p and modified hierarchical approximations are virtually identical, so results are only presented for bi-p spaces (of degrees 1 to 4). Computations are performed on uniform meshes and meshes graded near the singularity having 100, 225, 400, 625, and 900 elements. The graded meshes are uniform in the x_1 direction and have vertices in the

TABLE 4
Errors and effectivity indices for Example 1 using piecewise bi-p polynomial approximations on chevron-patterned meshes.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
100	0.342	0.888	0.301(−1)	0.897	0.221(−2)	0.797	0.145(−3)	0.909
225	0.245	0.932	0.156(−1)	0.895	0.868(−3)	0.875	0.438(−4)	0.971
400	0.187	0.936	0.857(−2)	0.864	0.338(−3)	0.927	0.125(−4)	0.922
625	0.153	0.950	0.578(−2)	0.861	0.191(−3)	0.945	0.591(−5)	0.933

x_2 direction at

$$x_{2,j} = (j/n)^{\beta}, \quad j = 0, 1, \cdots, n, \tag{5.5}$$

with $n = \sqrt{N}$. As suggested by Szabo and Babuska [12], we select $\beta = (p + 1/2)/(3/2 - 1/2)$ to match the singularity of the solution and recover the optimal $O(h^p)$ convergence rate in H^1 under h-refinement.

Exact errors and global effectivity indices in H^1 are presented in Tables 5 and 6 for uniform and graded meshes, respectively. A priori estimates indicate that the discretization error behaves as $O(h)$ on uniform meshes of spacing $h = 1/n$. A posteriori error estimates based on p-refinement, such as those of §3, would, therefore, not be expected to perform well under these conditions. The results of Table 5 confirm this. While effectivity indices for $p \neq 3$ are not bad, convergence under h-refinement is either non-existent or very slow. As shown in the upper portion of Fig. 2, poor results are due to large errors and local effectivity indices on elements adjacent to $x_2 = 0$. Results with highly-graded meshes (cf. Table 6 and the lower portion of Fig. 2) substantially improve performance. Global effectivity indices appear to converge to unity and local errors are closer to an equilibrated state.

Example 3. Consider a Dirichlet problem for Laplace's equation on a unit square with the data selected so that the exact solution in polar coordinates is

$$u(\mathbf{x}) = u(r,\phi) = r^{2/3}\sin(\frac{2}{3}\phi), \tag{5.6}$$

TABLE 5
Errors and effectivity indices for Example 2 using piecewise bi-p polynomial approximations on uniform meshes.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
100	0.480(−1)	0.875	0.665(−2)	0.988	0.298(−2)	0.373	0.161(−2)	0.824
225	0.333(−1)	0.885	0.443(−2)	0.988	0.199(−2)	0.373	0.107(−2)	0.824
400	0.256(−1)	0.891	0.333(−2)	0.988	0.149(−2)	0.373	0.806(−3)	0.824
625	0.209(−1)	0.896	0.266(−2)	0.988	0.119(−2)	0.373	0.645(−3)	0.824
900	0.177(−1)	0.899	0.222(−2)	0.988	0.993(−3)	0.373	0.537(−3)	0.824

TABLE 6
Errors and effectivity indices for Example 2 using piecewise bi-p polynomial approximations on highly-graded meshes.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
100	0.392(−1)	1.002	0.131(−2)	0.9957	0.129(−3)	1.380	0.200(−4)	1.031
225	0.263(−1)	1.001	0.592(−3)	0.9971	0.396(−4)	1.268	0.422(−5)	1.020
400	0.198(−1)	1.001	0.336(−3)	0.9979	0.170(−4)	1.208	0.138(−5)	1.015
625	0.158(−1)	1.000	0.216(−3)	0.9983	0.876(−5)	1.149	0.576(−6)	1.012
900	0.132(−1)	1.000	0.151(−3)	0.9987	0.510(−5)	1.126	0.281(−6)	1.009

which has a point singularity at the origin.

We initially solve this problem using piecewise bi-p approximations on uniform meshes having 100, 225, 400, and 625 elements. Exact errors and global effectivity indices are presented in Table 7. Local errors and effectivity indices on a 400-element mesh are presented in the upper portion of Fig. 4. As with Example 2, solutions on uniform meshes concentrate errors in the element adjacent to the singularity. The a posteriori error estimates cannot performs well under these conditions and poor global effectivity indices result.

Results are also obtained using piecewise bi-p and modified hierarchical approximations on locally graded meshes that were generated by refining the element closest to the singularity of a uniform N-element mesh. Refinement of the element nearest the singularity consists of generating n elements along each coordinate axis according to the distribu-

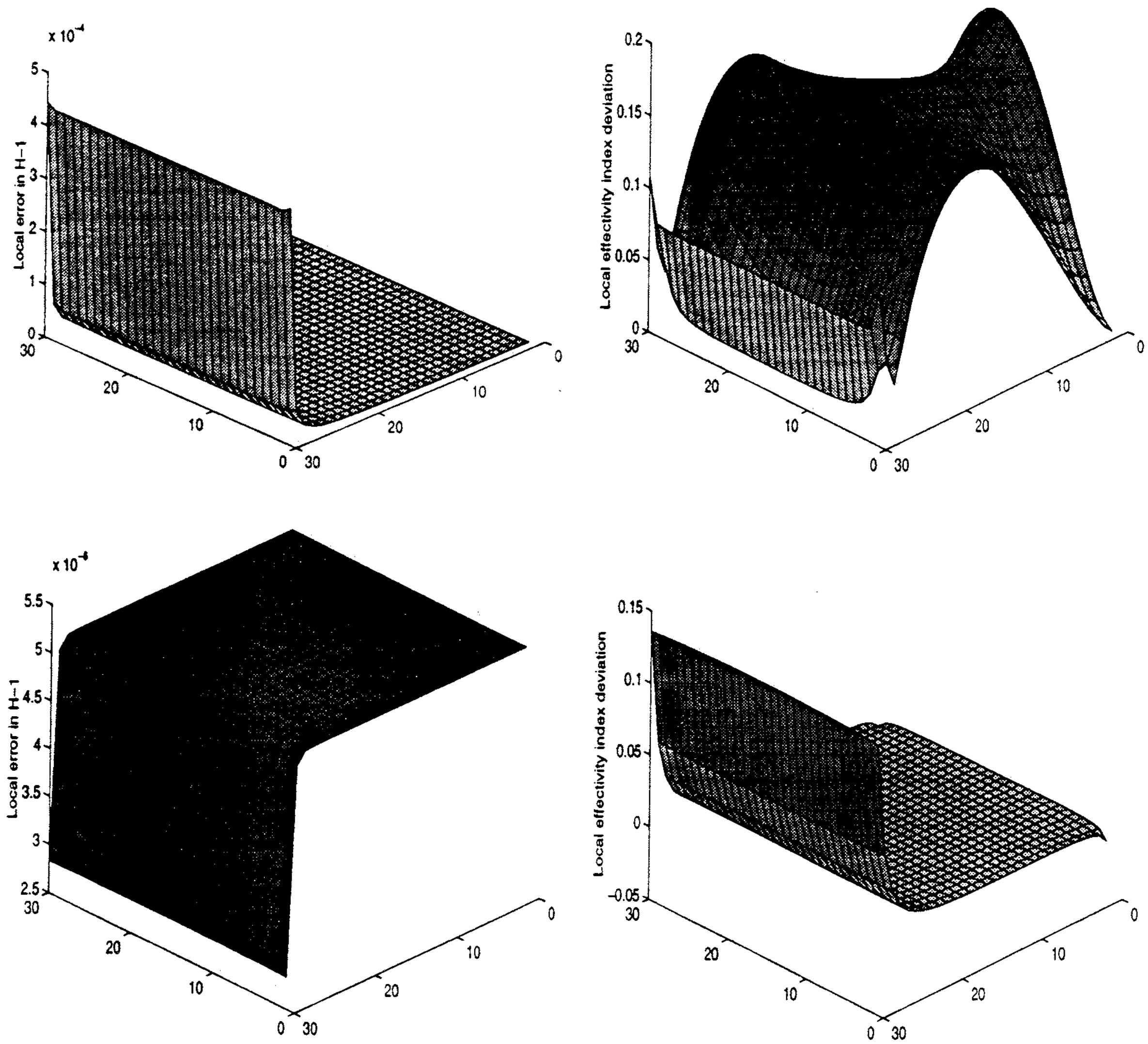

FIG. 2. *Local errors (upper-left) and the difference between the local effectivity indices and unity (upper-right) for Example 2 on a uniform 900-element mesh using piecewise bi-p approximations with $p = 2$. Similar data for computations performed on a highly-graded mesh are shown at the bottom.*

TABLE 7
Errors and effectivity indices for Example 3 using piecewise bi-p polynomial approximations on uniform meshes.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
100	0.466(−1)	0.6430	0.237(−1)	0.191	0.162(−1)	0.310	0.122(−1)	0.041
225	0.357(−1)	0.6482	0.181(−1)	0.191	0.124(−1)	0.311	0.929(−2)	0.041
400	0.296(−1)	0.6512	0.149(−1)	0.191	0.102(−1)	0.311	0.767(−2)	0.041
625	0.256(−1)	0.6531	0.129(−1)	0.191	0.882(−2)	0.311	0.660(−2)	0.041

tion (5.5) and generating a diagonal from the upper right vertex of the smallest square element to that of the original square element. As shown in Fig. 3 for $N = 25$ and $n = 4$, this process creates a mesh with N square and $2(n - 1)$ trapezoidal elements. Results in Tables 8 and 9 use the following combinations of N and n: 400, 10; 625, 15; 900, 20; and 1225, 50. Values of β are selected as 1/(2/3) for $p = 1$ and $(p/2)/(2/3)$ for $p > 1$. Local errors and effectivity indices are presented for piecewise bi-p approximations with $p = 4$ in the lower portion of Fig. 4.

The severe grading has reduced the local error on the element adjacent to the singularity and this has substantially improved the performance of the global and local effectivity indices. As with Examples 1 and 2, results for $p = 3$ are poorer than those for other orders. Effectivity indices are closer to unity with bi-p approximations than with modified hierarchical approximations. Additional equilibration of loading on the edges of odd-order approximations may be necessary to improve the performance of the error estimate [11]. A similar degradation of performance was observed by Babuska and Yu [5] with first- and second-order approximations in the presence of singularities.

Example 4. Consider the convection-diffusion problem

$$u_t - \Delta u + u_{x_1} + u_{x_2} = f(\mathbf{x},t), \quad \mathbf{x} \in [0,1] \times [0,1], \quad t > 0, \tag{5.7a}$$

with $f(\mathbf{x},t)$, the initial, and the Dirichlet boundary conditions specified so that the exact solution is

TABLE 8
Errors and effectivity indices for Example 3 using piecewise bi-p polynomial approximations on highly-graded meshes.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
418	0.161(−1)	1.082	0.137(−2)	0.960	0.272(−3)	1.986	0.420(−4)	1.076
653	0.138(−1)	1.055	0.113(−2)	0.999	0.227(−3)	1.811	0.343(−4)	1.105
938	0.123(−1)	1.045	0.974(−3)	1.023	0.198(−3)	1.712	0.298(−4)	1.113
1273	0.111(−1)	1.040	0.864(−3)	1.038	0.177(−3)	1.654	0.266(−4)	1.114

TABLE 9
Errors and effectivity indices for Example 3 using piecewise modified hierarchical polynomial approximations on highly-graded meshes.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
418	0.161(−1)	1.082	0.160(−2)	0.840	0.347(−3)	1.723	0.696(−4)	0.652
653	0.138(−1)	1.055	0.132(−2)	0.871	0.288(−3)	1.567	0.577(−4)	0.661
938	0.123(−1)	1.045	0.114(−2)	0.889	0.250(−3)	1.480	0.504(−4)	0.661
1273	0.111(−1)	1.040	0.101(−2)	0.900	0.223(−3)	1.429	0.451(−4)	0.660

$$u(x_1,x_2,t) = \tfrac{1}{2}[1 - \tanh(10x_1 + 2x_2 - 10t - 2)]. \tag{5.7b}$$

We solve this problem on $0 < t \le 0.5$ using uniform meshes having 100, 400, 900, and 1600 square elements with piecewise bi-p and modified hierarchical polynomial approximations of orders 1 to 4. Temporal integration utilizes the backward-difference software system DASSL [10] with error tolerances of 10^{-6} for $p = 1, 2$, 10^{-8} for $p = 3$, and 10^{-10} for $p = 4$. Such small tolerances minimize temporal discretization errors relative to the spatial errors that we are studying.

Exact errors and effectivity indices in H^1 obtained using (3.8, 14) at $t = 0.5$ are presented in Tables 10 and 11, respectively, for piecewise bi-p and modified hierarchical approximations. The results again indicate convergence of effectivity indices to unity under mesh refinement. The error estimates have a good range of applicability with effectivity indices in excess of 0.9 for virtually all computations. Once again, performance is poorer when $p = 3$. Results are slightly better with modified hierarchical than with bi-p approximations.

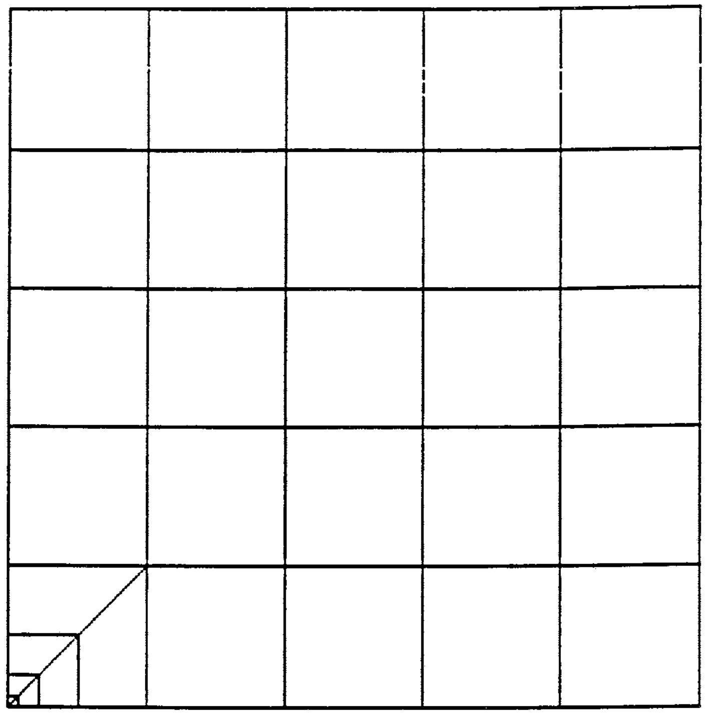

FIG. 3. *Structure of a highly-graded mesh with $N = 25$ and $n = 4$ for Example 3.*

Example 5. Consider the nonlinear reaction-diffusion equation

$$u_t - \tfrac{1}{2}\Delta u - qu^2(1 - u), \quad \mathbf{x} \in [0,1]\times[0,3/2], \quad t > 0, \tag{5.8a}$$

with $q \geq 0$ and the initial and Dirichlet boundary conditions specified so that the exact solution is

$$u(x_1,x_2) = [1 + e^{\sqrt{q/2}(x_1 + x_2 - t\sqrt{q/2})}]^{-1}. \tag{5.8b}$$

This solution represents a wave front moving normal to the line $x_1 = -x_2$ with speed $\sqrt{q/2}$.

We solve (5.8) with $q = 20$ on $0 < t \leq 0.5$ using the meshes and piecewise polynomial approximations specified in Example 4. Temporal tolerances are selected as 10^{-6} for $p = 1$, 10^{-10} for $p = 2$, and 10^{-13} for $p = 3, 4$.

Exact errors and effectivity indices at $t = 0.5$ appear in Tables 12 and 13, respectively for piecewise bi-p and modified hierarchical approximations. Results are

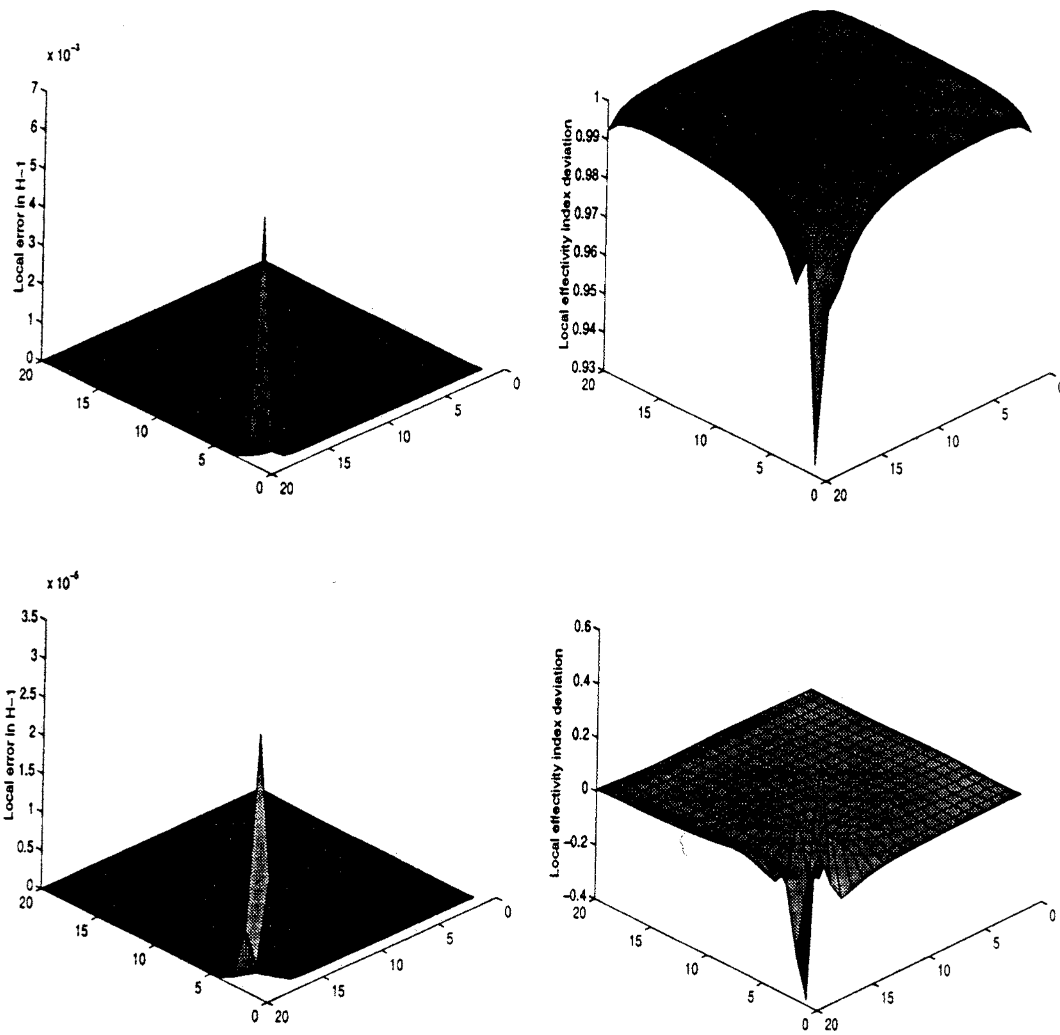

FIG. 4. *Local errors (upper-left) and the difference between the local effectivity indices and unity (upper-right) for Example 3 on a uniform 400-element mesh using piecewise bi-p approximations with $p = 4$. Similar data for computations performed on a highly-graded 418-element mesh are shown at the bottom.*

TABLE 10
Errors and effectivity indices for Example 4 using piecewise bi-p polynomial approximations.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
100	0.478	0.739	0.101	0.885	0.192(−1)	0.426	0.356(−2)	0.920
400	0.235	0.928	0.255(−1)	0.970	0.252(−2)	0.754	0.238(−3)	0.979
900	0.157	0.968	0.114(−1)	0.987	0.756(−3)	0.880	0.477(−4)	0.991
1600	0.118	0.981	0.642(−2)	0.992	0.320(−3)	0.930	0.152(−4)	0.995

TABLE 11
Errors and effectivity indices for Example 4 using piecewise modified hierarchical polynomial approximations.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
100	0.380	0.858	0.957(−1)	0.972	0.186(−1)	0.556	0.350(−2)	0.970
400	0.195	0.956	0.252(−1)	0.990	0.251(−2)	0.848	0.237(−3)	0.992
900	0.131	0.980	0.113(−1)	0.995	0.753(−3)	0.930	0.476(−4)	0.996
1600	0.984(−1)	0.989	0.640(−2)	0.997	0.319(−3)	0.960	0.152(−4)	0.998

comparable to those of Example 4, except that effectivity indices for $p = 3$ are much closer to unity here than there.

6. Discussion. We show that a posteriori estimates of spatial discretization errors of piecewise bi-p polynomial finite element solutions of two-dimensional elliptic and parabolic problems [2, 3, 5, 14, 15] extend to other spaces of order p. The finite element space must contain all monomial terms of degree $p + 1$ except the principal terms x_1^{p+1} and x_2^{p+1}. If so, then estimates involving jumps in solution gradients at element vertices when p is odd and from the solution of local elliptic or parabolic problems when p is even are asymptotically correct on rectangular-element grids. The error estimates are stated for arbitrarily structured grids of quadrilateral elements and computational results of §5 show that they may be asymptotically correct in situations that are not supported by the present theory.

The error estimates are simple to construct and require at most nearest-neighbor

TABLE 12
Errors and effectivity indices for Example 5 using piecewise bi-p polynomial approximations.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
100	0.278(−1)	0.949	0.924(−3)	0.995	0.295(−4)	0.920	0.898(−6)	0.999
400	0.137(−1)	0.977	0.231(−3)	0.999	0.369(−5)	0.966	0.562(−7)	1.000
900	0.909(−2)	0.985	0.102(−3)	0.999	0.109(−5)	0.979	0.111(−7)	1.000
1600	0.681(−2)	0.989	0.577(−4)	1.000	0.462(−6)	0.979	0.351(−8)	1.000

TABLE 13
Errors and effectivity indices for Example 5 using piecewise modified hierarchical polynomial approximations.

p	1		2		3		4	
N	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ	$\|e\|_1$	θ
100	0.446(−1)	0.985	0.226(−1)	0.967	0.107(−3)	1.106	0.530(−5)	0.867
400	0.219(−1)	0.996	0.561(−2)	0.990	0.128(−4)	1.031	0.297(−6)	0.961
900	0.146(−1)	0.998	0.249(−3)	0.996	0.377(−5)	1.014	0.574(−7)	0.982
1600	0.114(−1)	0.999	0.140(−3)	0.998	0.160(−5)	1.008	0.180(−7)	0.990

information from the finite element solution; hence, they are efficient for both serial and parallel computation. Temporal superconvergence appears to be robust; thus, there is little advantage of using the parabolic error estimation procedure relative to the elliptic procedure for even-order approximations. An exception might occur when using error estimates to control mesh motion (r-refinement).

Focusing on spatial error estimation, we ensured that temporal errors were negligible relative to spatial errors. In a practical computational system, however, the temporal and spatial errors must be related. One way of doing this is to maintain the local temporal error per step at a small percentage of the total error [7, 9].

Several theoretical extensions of the error estimation procedures described herein are necessary. For example, the performance of the error estimates in the presence of singularities and singular perturbations needs investigation. The latter situation involves small diffusivities [1] whereas the error estimates rely on diffusion dominance. At the very least, error estimates become less robust near singularities and in the presence of singular perturbations. A theory is needed for nonlinear problems and vector systems. Likewise, convergence analyses are needed on meshes of arbitrary triangular and quadrilateral elements. The error estimates can be easily extended to three-dimensional cubic elements and the present theory holds in this case. However, analyses are needed for meshes of tetrahedral and hexahedral elements.

REFERENCES

[1] S. Adjerid, M. Aiffa, and J.E. Flaherty, *High-order finite element methods for singularly-perturbed elliptic and parabolic problems,* SIAM J. Appl. Math. 55 (1995) pp. 520-543.

[2] S. Adjerid, I. Babuska, and J.E. Flaherty, *A posteriori error estimation for the finite element method of lines of parabolic systems,* 1996, in preparation.

[3] S. Adjerid, J.E. Flaherty, and Y.J. Wang, *A posteriori error estimation with finite element methods of lines for one-dimensional parabolic systems,* Numer. Math., 65 (1993), pp. 1-21.

[4] M. Ainsworth and J. T. Oden, *A unified approach to a posteriori error estimation using element residual methods,* Numer. Math., 65 (1993), pp. 23-50.

[5] I. Babuska and D. Yu, *Asymptotically exact a-posteriori error estimator for biquadratic elements,* Finite Elem. Anal. Des., 3 (1987), pp. 341-354.

[6] R.E. Bank and A. Weiser, *Some a posteriori error estimators for elliptic partial differential equations,* Maths. Comp., 9 (1985), pp. 283-302.

[7] M. Berzins, *Global error estimation in the method of lines for parabolic equations,* SIAM J. Sci. Stat. Comput., 9 (1988), pp. 687-703.

[8] P. Carnevali, R.B. Morris, Y. Tsuji, and G. Taylor, *New basis functions and computational procedures for p-version finite element analysis,* Int. J. Numer. Meth. Engng., 36 (1993), pp. 3759-3779.

[9] J.E. Flaherty and P.K. Moore, *Integrated space-time adaptive hp-refinement methods for parabolic systems,* Appl. Numer. Maths., 16 (1995), pp. 317-341.

[10] L.R. Petzold, *A description of DASSL: a differential/algebraic system solver,* Rep. Sand. 82-8637, Sandia National Laboratory, Livermore, 1982.

[11] T. Strouboulis and K.A. Haque, *Recent experiences with error estimation and adaptivity, Part I: Review of error estimators for scalar elliptic problems,* Comput. Meths. Appl. Mech. Engng., 97 (1992), pp. 399-436.

[12] B.A. Szabo and I. Babuska, *Introduction to Finite Element Analysis,* J. Wiley and Sons, New York, 1991.

[13] R. Verfürth, *A posteriori error estimation and adaptive mesh-refinement techniques,* J. Comput. Appl. Math., 1996, to appear.

[14] D. Yu, *Asymptotically exact a-posteriori error estimator for elements of bi-even degree,* Mathematica Numerica Sinica, 13 (1991), pp. 89-101.

[15] D. Yu, *Asymptotically exact a-posteriori error estimator for elements of bi-odd degree,* Mathematica Numerica Sinica, 13 (1991), pp. 307-314.

[16] O.C. Zienkiewicz, *The Finite Element Method,* 3rd Ed., McGraw-Hill, New York, 1977