Additive Schwarz Algorithms for Solving hp-Version Finite Element Systems on Triangular Meshes

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Abstract

Highly parallelizable domain decomposition Dirichlet-Dirichlet solvers for *hp*-version finite element methods on angular quasiuniform triangular meshes are studied under different assumptions on a reference element. The edge coordinate functions of a reference element are allowed to be either nodal with special choices of nodes, or hierarchical polynomials of several types. These coordinate functions are defined within elements as being arbitrary or discrete quasi-harmonic coordinate functions. The latter are obtained from explicit and inexpensive prolongation operators. In all situations, we are able to suggest preconditioners which are spectrally equivalent to the global stiffness matrix, which only require element-by-element and edge-by-edge operations, and which reduce computational cost. In this way, elimination is avoided when dealing with the interface problem. The domain decomposition algorithms essentially use prolongation operators from the interface boundary inside the subdomains of the decomposition according to the approach initially used for the *hp*-version finite element methods with quadrilateral elements [12]–[15].

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1 Introduction

The *hp*-version of the finite element method (FEM) is well suited to efficient computation in many practical situations. When used adaptively, it is capable of producing exponential rates of convergence [4, 31, 36]. It also permits the use of highly parallel domain decomposition (*DD*) solution procedures.

Indeed, an important consideration in the efficient implementation of *hp*-version FEMs is the algebraic solver and, for this reason, the development of *hp*-version *DD* solvers has received a great deal of recent attention. The basic ideas arise from studies of *DD* algorithms in general and for the *h*-version FEM. Fundamental developments are due to Bramble *et al.* [7, 8], Nepomnyaschikh [28], Widlund [35], and Lions [24]. The first analysis of *DD* Dirichlet-Dirichlet solution algorithms for the *p*-version FEM was done by Babuška *et al.* [3], who also developed important technical tools for future studies. Assuming orthogonality of the internal and edge coordinate functions of elements, they obtained an $\mathcal{O}(1+\log^2 p)$ estimate of the relative condition number similar to the $\mathcal{O}((H/h)^2)$ estimate of Bramble *et al.* [7] for the *h*-version FEM. Further developments of *DD* Dirichlet-Dirichlet algorithms, which are also our concern, were aimed at obtaining preconditioners that are spectrally equivalent or almost spectrally equivalent to the stiffness matrix [1] and providing lower computational cost. Cost was reduced by using less expensive solvers for the internal and interface subproblems and avoiding direct elimination procedures. Ivanov and Korneev [13]–[15] considered *hp*-version FEMs on square reference elements with products of integrated Legendre polynomials used for coordinate functions. They showed that a spectrally equivalent preconditioner for internal problems is a matrix of a five-point finite-difference-like operator of second order in the reference square. The component of the *DD* preconditioner related to the interface problem was obtained by an efficient Schur complement preconditioning and the use of explicit prolongation operators. Some alternative preconditioners were suggested by Korneev and Jensen [22, 23] who found the relative condition number in the range $\mathcal{O}(1)-\mathcal{O}(1+\log^3 p)$.

The problem of inexact solvers on interface surfaces for the *hp*-version FEM was considered by Ainsworth [2], who generalized the result of Bramble *et al.* [7] to the *hp*-version, and Guo and Cao [17]. Their approach relies on a transformation of the basis in the space of polynomial traces on the edge of the reference element to a basis of Chebyshev polynomials. This was used for similar purpose in a slightly different algorithmic setting by Ivanov and Korneev [12]–[15] and Korneev and Jensen [22]. Iterative solvers for adaptive *hp*-version finite element discretizations based on two-level orthogonalization and coarse grid preconditioning was considered by Oden *et al.* [29]. Pavarino [30], Widlund [34], and Casarin [10], respectively, studied *DD* preconditions with overlapping regions, for *hp*-version mortar models, and for the nodal *hp*-version FEM. The approach to preconditioning initiated by Babuška *et al.* [3] is based on orthogonalization on the reference element and has been developed by Mandel [26] (and the references therein). There are also a number of studies on *DD* solvers for special types of partial differential equations, for models based on spectral elements, and for mixed formulations.

[1] In this paper, a *preconditioner* for a symmetric nonnegative matrix $\mathbf{A}$ is a symmetric nonnegative matrix $\mathbf{B}$ such that $\ker(\mathbf{B}) \subset \ker(\mathbf{A})$ and $\mathrm{cond}(\mathbf{B}^+\mathbf{A}) < \mathrm{cond}(\mathbf{A})$, where $\mathrm{cond}(\mathbf{A})$ is the ratio of the maximal eigenvalue to the minimal nonzero eigenvalue of $\mathbf{A}$ and $\mathbf{B}^+$ is the pseudo-inverse of $\mathbf{B}$. The *relative or generalized condition number* of $\mathbf{A}$ is $\mathrm{cond}(\mathbf{B}^+\mathbf{A})$.

The results of this paper, as those of Ivanov and Korneev [12]–[15]and Korneev and Jensen [22, 23], rely upon a specific transformation of the edge and vertex functions to produce efficient preconditionings of the global stiffness matrix by using its block-diagonal part with independent blocks related to the interior unknowns of each subdomain of decomposition and to the interface unknowns. The transformation corresponds to the prolongation of the traces of finite element functions on the inter-element boundary inside elements as discrete quasi-harmonic functions. If the edge and vertex unknowns are not split further, then the resulting preconditioner is spectrally equivalent to the global stiffness matrix. Further formal splitting of the edge and vertex unknowns may introduce an additional factor of $\ln(1+p)$ in the relative condition number depending on the definition of the vertex coordinate functions. We further reduce computational cost by obtaining preconditioners for the diagonal blocks of the global stiffness matrix. This makes it possible to replace expensive direct solutions at each leading iteration of the interior and interface problems with far less costly direct or secondary iterative procedures. By this, we arrive at *DD* preconditioners that are either spectrally equivalent or almost spectrally equivalent to the global stiffness matrix and, hence, are easily solved

Our primary concerns are establishing some basic facts related to the general form of *DD* preconditioners and analyzing several ways of preconditioning interface problems. We also provide simpler analyses for some basic results on *DD* preconditioning. Optimizing the preconditioners for the interior problems on the subdomains of the decomposition is, of course, a much more difficult problem for triangular elements than for quadrilaterals. Herein, we do not provide an ultimate solution and suggest only initial steps in this direction.

The procedure for defining edge and vertex coordinate functions by means of prolongation operators may be easily incorporated into existing software. For this reason, Shephard *et al.* [33] used a description of edge and face functions based on continuation of the traces through blending functions. In this connection, we distinguish two major cases: (*i*) discrete quasi-harmonic edge coordinate functions and (*ii*) arbitrary edge coordinate functions. The underlying ideas of preconditioning are more transparent in Case (*i*) as well as the formulation and analyses of the algorithms. The suggestions that are useful for Case (*i*) may also be used for Case (*ii*). By these reasons, Case (i) is paid enough attention. But the authors, as the estimates of computational work given in this paper, don't imply that among several approaches to *DD* preconditioning , the transformation to the basis (i) for the preliminary step will be generally, *e.g.*, for all p, the best choice. To the best of the authors knowledge, it was never used in *DD* algorithms for h-version FEMs.

We consider algorithms where elimination of internal unknowns is replaced by less expensive procedures; however, elimination of internal unknowns is still used in practice. As we show, all of our preconditioners that have been designed using Case (*i*) for the interface sub-problem are equally applicable in the general Case (*ii*) to systems obtained upon elimination of internal unknowns.

The paper is organized as follows. In Section 2, we formulate the problem and describe the finite element space. Section 3 presents reference elements with the discrete quasi-harmonic edge coordinate functions, which are specified in the reference element by means of prolongation operators. In Section 4, we consider several *DD* preconditioners depending on the type of the edge coordinate functions and the way of handling vertex unknowns, which in a preconditioner may or may not be split from the rest of the system. Preconditioning in the case of arbitrary

coordinate functions is studied in Section 5. Sections 4 and 5 contain estimates of the generalized condition numbers related to our preconditioners. Technical proofs are given in Section 6.

2 Finite Element Space

For definiteness, consider a Dirichlet problem for a second-order elliptic partial differential equation in a two-dimensional domain Ω. The generalized formulation of this problem involves determining $u \in V$ such that

$$a_\Omega(u,v) - (f,v)_\Omega = 0\,, \qquad \text{for all } v \in V\,, \tag{2.1}$$

where the bilinear form $a_\Omega(u,v)$ is symmetric, bounded on $V \times V$, and V-elliptic with V being the space $\overset{0}{H}_1(\Omega)$. [2] Thus, there exist constants $\mu_1, \mu_2 > 0$ such that

$$\mu_1 ||v||^2_{1,\Omega} \le a_\Omega(v,v)\,, \qquad a_\Omega(v,w) \le \mu_2 ||v||_{1,\Omega} ||w||_{1,\Omega}\,, \tag{2.2}$$

The problem is discretized into an assemblage of finite elements τ_r, $r = 1,2,\ldots,\mathcal{R}$, specified by nondegenerate compatible mappings $\mathbf{x} = \mathcal{X}^{(r)}(\mathbf{y}) : \bar{\tau}_0 \to \bar{\tau}_r$ having positive Jacobians. We assume that $\bar{\tau}_{r_1} \cap \bar{\tau}_{r_2} \in \emptyset$, $r_1 \ne r_2$, unless $\bar{\tau}_{r_1}$, $\bar{\tau}_{r_2}$ have a vertex or an edge in common. Such an assembly of triangles is called a *triangulation* and is denoted as S. The reference triangle may either be the unit right triangle $\tau_0 = \{\mathbf{y} : \ 0 < y_1, y_2, (y_1 + y_2) < 1\}$ or the unilateral triangle with vertices $\mathbf{y} = (-1,0), (1,0), (0,\sqrt{3})$. When elements have curved edges, the mappings $\mathcal{X}^{(r)}$ are nonlinear and may be characterized in different ways. For this purpose, we introduce the Lamé coefficients

$$H_k^{(r)} = [\sum_{l=1}^{2} (\partial \mathcal{X}_l^{(r)} / \partial y_k)^2]^{1/2}, \qquad k = 1,2\,.$$

Also, let $\mathbf{i}_k(\mathbf{y})$ be a unit vector in the $\mathbf{x}$-plane tangent to the line y_{3-k} = constant and directed towards increasing y_k, $k = 1,2$. We introduce $\theta^{(r)}(\mathbf{y})$ as the angle between $\mathbf{i}_1(\mathbf{y})$ and $\mathbf{i}_2(\mathbf{y})$. Then the *generalized quasiuniform mesh* conditions are expressed as

$$0 < \alpha^{(1)} h \le H_k^{(r)} \le h\,, \qquad 0 < \theta \le \theta^{(r)} \le \pi - 2\theta\,, \quad \alpha^{(1)}, \theta = \text{constant} \tag{2.3}$$

$$|D_y^q \mathcal{X}^{(r)}(\mathbf{y})| \le c h^{|q|}\,, \qquad 1 \le |q| \le n\,, \qquad \mathbf{y} \in \tau_0, \qquad r = 1,2,\ldots,\mathcal{R}, \tag{2.4}$$

where h is a mesh spacing parameter. The inequalities (2.4) are needed to analyze convergence, but the study of *DD* preconditioners only requires satisfaction of (2.3). If, however, (2.4) is satisfied for, at least, $n = 2$, then, with h sufficiently small, (2.3) may be replaced by the geometric conditions [20]

$$\bar{\alpha}^{(1)} h \le \rho_{r,2} \le h\,, \qquad \bar{\theta} \le \rho_{r,1}/\rho_{r,2}\,, \tag{2.5}$$

where $\rho_{r,1}$ and $\rho_{r,2}$ are, respectively, the radii of the largest inscribed and smallest circumscribed circles relative to τ_r. In this case, (2.3) may also be understood in a simple geometric sense as restrictions on the lengths of edges and angles of the triangle τ_r' having vertices common with the triangle τ_r and straight sides. Equations (2.3) and (2.5) are called the conditions of *generalized angular quasiuniformity*, if $h = h(r)$.

[2] Fundamental notation is summarized in Section 6.3.

Lemma 2.1. *Let Ω be an arbitrary domain with boundary $\partial\Omega \in C^{(t)}$, $t \geq 2$, and $h > 0$ be sufficiently small. Then there exists an exact triangulation S of Ω having compatible mappings $\mathcal{X}^{(r)} : \bar{\tau}_0 \to \bar{\tau}_r$ and satisfying (2.3), (2.4) with $\alpha^{(1)} = \alpha^{(1)}(\Omega)$, $\theta = \theta(\Omega)$, and $n = (t-1)$.*

Proof. See Korneev [19, 20]. □

When solving (2.1), polynomial mappings, particularly iso-parametric ones, may be used for $\mathcal{X}^{(r)}$, and S need not exactly coincide with Ω. However, since convergence of approximate solutions is not our concern, only S matters in this study. Thus, unless specifically noted, we assume that Ω is the domain of some triangulation S and that the generalized quasiuniformity conditions are fulfilled. For simplicity, we also assume that only one reference element is used for the finite element assembly.

Let the reference element coordinate function be denoted as $p_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ taking values on some set ω_Δ. In the case under consideration,

$$\mathrm{span}_{\boldsymbol{\alpha} \in \omega_\Delta} p_{\boldsymbol{\alpha}} = \mathcal{P}_{p,x} \,,$$

and the coordinate function $p^{(r)}_{\boldsymbol{\alpha}}$ of an element $\mathcal{E}_r$ defined on τ_r satisfies

$$p^{(r)}_{\boldsymbol{\alpha}}(\mathcal{X}^{(r)}(\mathbf{y})) = p_{\boldsymbol{\alpha}}(\mathbf{y}), \qquad \forall\, \mathbf{y} \in \tau_0 \,. \tag{2.6}$$

These functions induce the following spaces on τ_r and Ω:

$$\begin{aligned} \mathcal{H}(\tau_r) &= \mathrm{span}_{\boldsymbol{\alpha} \in \omega_\Delta} p^{(r)}_{\boldsymbol{\alpha}} \,, \quad \mathcal{H}(\Omega) = \{v \in C(\bar{\Omega}) : \; v|_{\tau_r} \in \mathcal{H}(\tau_r),\; r = 1, 2, \ldots, \mathcal{R}\} \,, \\ \mathcal{H}^0(\Omega) &= \mathcal{H}(\Omega) \cap V \,. \end{aligned}$$

The system of finite element algebraic equations

$$\mathbf{K}\mathbf{u} = \mathbf{f} \tag{2.7}$$

is equivalent to (2.1) in $\mathcal{H}^0(\Omega)$, so that

$$(\mathbf{K}\mathbf{w}, \mathbf{v}) = a_\Omega(w, v), \qquad (\mathbf{f}, \mathbf{v}) = (f, v)_\Omega \,, \tag{2.8}$$

for any vectors $\mathbf{w}, \mathbf{v}$ and the corresponding functions $w, v \in \mathcal{H}^0(\Omega)$.

3 Reference Elements

The choice of reference element coordinate functions critically affects the properties of the stiffness matrix $\mathbf{K}$. We consider several choices that differ in their edge and vertex coordinate functions. The traces of the coordinate functions of one of these choices on each edge of τ_0 are Lagrange interpolation polynomials relative to sets of nodes that are the same as those used by Ivanov and Korneev [12] on the edges of square reference elements. We call such coordinate functions "nodal," although they may not be nodal within τ_0. For other possibilities, the traces of edge coordinate functions on $\partial\tau_0$ are hierarchical polynomials and the vertex coordinate functions are linear. We need not select specific internal coordinate functions because we neither consider fast solvers for the internal subproblems nor do our main results depend on the specific form of the internal coordinate functions; however, different choices may produce better conditioning of the stiffness matrix [36, 9] .

3.1 Nodal Edge and Vertex Coordinate Functions

Let $V^{(k)}$ denote the vertices of the reference triangle ordered in a counterclockwise direction, $T^{(k)}$ be the edge joining vertices $V^{(k-1)}$, $V^{(k)}$, and $s_k = L_k - L_{k-1}$, where L_k are barycentric coordinates. The indices k are understood to be defined modulo 3. With ω_Δ being the set of indices $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, let

- $\boldsymbol{\alpha} = (k, 0, 0)$, $k = 1, 2, 3$, identify the vertex coordinate functions,
- $\boldsymbol{\alpha} = (k, l, 0)$, $l = 1, 2, \ldots, p-1$, identify the edge coordinate functions associated with edge $T^{(k)}$ and
- $\boldsymbol{\alpha} \in \omega_{\Delta,I} := \{\boldsymbol{\alpha} : \ 1 \le \alpha_1, \alpha_2, (\alpha_1 + \alpha_2) \le p-1, \ \alpha_3 = 1\}$ identify the internal coordinate functions. For definiteness, we assume

$$p_{\boldsymbol{\alpha}} = L_1^{\alpha_1} L_2^{\alpha_2} L_3 \, . \tag{3.1}$$

In order to define the vertex and edge coordinate functions, let their traces on $\partial\tau_0$ be interpolation polynomials and continue them into τ_0 by a special prolongation operator. To this end, introduce the nodes

$$s_k^{(l)} = \cos(p-l)\pi p^{-1} \, , \qquad l = 0, 1, \ldots, p \, , \tag{3.2}$$

on $T^{(k)}$ and the corresponding Lagrange basis of p-degree polynomials

$$\phi^{(l)}(s_k) = \prod_{m=0, m\neq l}^{p} (s_k - s_k^{(m)})/(s_k^{(l)} - s_k^{(m)}), \qquad k = 1, 2, 3, \qquad l = 1, 2, \ldots, p-1 \, . \tag{3.3}$$

The traces of the edge coordinate functions are

$$p_{(k,l,0)}|_{T^{(i)}} = \begin{cases} \phi^{(l)}(s_i), & \text{if } i = k \\ 0, & \text{otherwise} \end{cases} \, , \qquad k = 1, 2, 3 \, , \qquad l = 1, 2, \ldots p-1 \, . \tag{3.4}$$

The traces of the vertex coordinate functions are

$$p_{(k,0,0)}|_{T^{(i)}} = \begin{cases} \phi^{(p)}(s_i) \, , & \text{if } i = k \\ \phi^{(0)}(s_i) \, , & \text{if } i = k+1 \\ 0 \, , & \text{otherwise} \end{cases} \, . \tag{3.5}$$

Suppose that v_0 is any function that is continuous on $\partial\tau_0$ and a polynomial of degree p or less on each edge $T^{(k)}$, $k = 1, 2, 3$. Let $I\!P_\tau$ be a linear prolongation operator, such that $v = I\!P_\tau v_0$ satisfies

$$||v||_{1,\tau_0} \le c_{I\!P} ||v_0||_{1/2,\partial\tau_0} \, , \quad |v|_{1,\tau_0} \le c_{I\!P} |v_0|_{1/2,\partial\tau_0} \, , \tag{3.6}$$

with $c_{I\!P}$ independent of p and v. With this prolongation, the vertex and edge coordinate functions are

$$p_{\boldsymbol{\alpha}} = I\!P_\tau p_{\boldsymbol{\alpha},\partial\tau} \, , \tag{3.7}$$

where $p_{\boldsymbol{\alpha},\partial\tau}$ are the traces of the coordinate functions $p_{\boldsymbol{\alpha}}$ on $\partial\tau_0$ given by (3.4),(3.5). The prolongation operators that may be used in computation are described in Section 3.3.

3.2 Hierarchical Vertex and Edge Coordinate Functions

The vertex coordinate functions for a hierarchical basis are the linear polynomials

$$p_{\boldsymbol{\alpha}} = L_{\alpha_1}, \qquad \boldsymbol{\alpha} = (\alpha_1, 0, 0), \qquad \alpha_1 = 1, 2, 3\,. \tag{3.8}$$

The traces of the edge coordinate functions on τ_0 may be defined by different sets of polynomials. One choice is

$$p_{\boldsymbol{\alpha}}|_{T^{(k)}} = \begin{cases} \tilde{L}_{l+1}(s_k)\,, & \text{if } k = \alpha_1 \\ 0\,, & \text{otherwise} \end{cases}, \qquad k = 1, 2, 3, \qquad l = 1, 2, \ldots, p-1\,, \tag{3.9a}$$

$$\tilde{L}_j = \beta_j \int_{-1}^{s_k} P_{j-1}(t)dt = \frac{\beta_j}{2j-1}[P_j(s_k) - P_{j-2}(s_k)]\,, \tag{3.9b}$$

with $P_j(s)$ being the Legendre polynomial of degree j and β_j being a normalization parameter. As a second choice, consider the edge coordinate functions with the traces

$$p_{\boldsymbol{\alpha}}|_{T^{(k)}} = \begin{cases} (1-s_k^2)s_k^{l-1}\,, & \text{if } k = \alpha_1 \\ 0\,, & \text{otherwise} \end{cases}, \qquad k = 1, 2, 3, \qquad l = 1, 2, \ldots, p-1\,. \tag{3.10}$$

Again, we use the prolongations (3.7) to extend traces inside the triangle.

3.3 Prolongation Operators

Construction of prolongation operators in polynomial spaces for which (3.6) holds were studied by Babuška and Suri [4], Babuška *et al.* [3], Bernardi *et al.* [5], and Munõz-Sola [27]. Since we use prolongation operators in our algorithms, we follow Bernardi *et al.* [5] where these operators have a simpler form for numerical realization.

Consider the equilateral triangle τ_0 described in Section 2 with $V^{(1)} = (-1, 0)$ for definiteness. To define edge functions, it suffices to consider one edge and we focus on $T^{(2)}$. Let $\psi(s_2)$, $s_2 \in [-1, 1]$, be a polynomial of degree p or less. If $\psi(s_2)$ is a trace of an edge function, it has the form

$$\psi(s_2) = (1 - s_2^2)\psi_0(s_2)\,,$$

where $\psi_0(s_2)$ is a polynomial of degree $p-2$ or less. We set

$$I\!P_\tau \psi := F_{2,0}(\psi)(\mathbf{x}) := 4L_1L_2F_2(\psi_0(s_2))(\mathbf{x})\,, \tag{3.11}$$

$$F_2(\chi)(\mathbf{x}) = \frac{\sqrt{3}}{2x_2}\int_{x_1-x_2/\sqrt{3}}^{x_1+x_2/\sqrt{3}} \chi(t)dt = \frac{1}{2}\int_{-1}^{1} \chi(x_1 + \frac{x_2}{\sqrt{3}}s)ds\,. \tag{3.12}$$

Liftings from the other edges are defined similarly; thus, we may write the edge coordinate functions $p_{\boldsymbol{\alpha}}$ for any $k = 1, 2, 3$ as

$$p_{\boldsymbol{\alpha}}(\mathbf{x}) = I\!P_\tau p_{\boldsymbol{\alpha},\partial\tau} := F_{k,0}(p_{\boldsymbol{\alpha},\partial\tau}|_{T^{(k)}})(\mathbf{x}) := L_{k-1}L_kF_k((1-s_k^2)^{-1}(p_{\boldsymbol{\alpha},\partial\tau}|_{T^{(k)}}))(\mathbf{x})\,. \tag{3.13}$$

It follows from (3.11) - (3.13) that

$$F_2(\chi(s_2))(\mathbf{x})|_{T^{(2)}} = \chi(s_2), \qquad F_{2,0}(\psi(s_2))(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \in \partial\tau_0 \setminus T^{(2)} \\ \psi(s_2), & \text{if } \mathbf{x} \in T^{(2)} \end{cases},$$

with similar expressions holding for F_k and $F_{k,0}$, $k = 1, 3$. Therefore, $(I\!P_\tau p_{\boldsymbol{\alpha},\partial\tau})|_{\partial\tau_0} = p_{\boldsymbol{\alpha},\partial\tau}$, and it is easy to see that $I\!P_\tau p_{\boldsymbol{\alpha},\partial\tau} \in \mathcal{P}_{p,\mathbf{x}}$.

With a goal of specifying the prolongations for the vertex coordinate functions, let

$$p_{\boldsymbol{\alpha},l}(s_l) = p_{\boldsymbol{\alpha}}|_{T^{(l)}} ,$$

where the traces are described, *e.g.*, by (3.5). We introduce the lifting

$$F_{k,\beta}(\psi(s_k)) := 2L_{k+(\beta-1)/2}F_k((1-\beta s_k)^{-1}\psi(s_k))(\mathbf{x})$$

for polynomials $\psi(s_k)$ that are specified on $T^{(k)}$ and which vanish at one of the ends, *i.e.*, $\psi(\beta) = 0$, and $\beta = -1$ or 1. With $\boldsymbol{\alpha} = (k, 0, 0)$, the function

$$\Psi_k(\mathbf{x}) = p_{\boldsymbol{\alpha},\partial\tau}(\mathbf{x}) - F_{k,-1}(p_{\boldsymbol{\alpha},k}(s_k))(\mathbf{x})$$

is continuous on $\partial\tau_0$ and only nonzero on side $T^{(k+1)}$. Thus, we may set

$$\begin{aligned} p_{\boldsymbol{\alpha}} = I\!P_\tau p_{\boldsymbol{\alpha},\partial\tau} &:= \hat{F}_k^{(k)}(p_{\boldsymbol{\alpha},\partial\tau})(\mathbf{x}) , \\ \hat{F}_k^{(k)}(p_{\boldsymbol{\alpha},\partial\tau})(\mathbf{x}) &:= F_{k,-1}(p_{\boldsymbol{\alpha},k}(s_k))(\mathbf{x}) + F_{k+1,0}(\Psi_k|_{T^{(k+1)}})(\mathbf{x}) , \end{aligned} \tag{3.14}$$

and it is easy to verify that $\hat{F}_k^{(k)}(p_{\boldsymbol{\alpha},\partial\tau})(\mathbf{x}) \in \mathcal{P}_{p,\mathbf{x}}$ and $\hat{F}^{(k)}(p_{\boldsymbol{\alpha},\partial\tau})(\mathbf{x})|_{\partial\tau_0} = p_{\boldsymbol{\alpha},\partial\tau}$.

Another possible lifting is defined as

$$\begin{aligned} I\!P_\tau p_{\boldsymbol{\alpha},\partial\tau} &:= \hat{F}_k^{(k+1)}(p_{\boldsymbol{\alpha},\partial\tau})(\mathbf{x}) , \\ \hat{F}_k^{(k+1)}(p_{\boldsymbol{\alpha},\partial\tau})(\mathbf{x}) &:= F_{k+1,1}(p_{\boldsymbol{\alpha},k+1}(s_{k+1}))(\mathbf{x}) + F_{k,0}(\Psi_{k+1}|_{T^{(k)}})(\mathbf{x}) , \end{aligned} \tag{3.15}$$

where

$$\Psi_{k+1}(\mathbf{x}) = p_{\boldsymbol{\alpha},\partial\tau}(\mathbf{x}) - F_{k+1,1}(p_{\boldsymbol{\alpha},k+1}(s_{k+1}))(\mathbf{x}) .$$

In order to make the liftings independent of the direction of the vertex indexing, we define them as

$$I\!P_\tau p_{\boldsymbol{\alpha},\partial\tau} := \frac{1}{2}(\hat{F}_k^{(k)}(p_{\boldsymbol{\alpha},\partial\tau})(\mathbf{x}) + \hat{F}_k^{(k+1)}(p_{\boldsymbol{\alpha},\partial\tau})(\mathbf{x})) . \tag{3.16}$$

However, the lifting (3.16) requires twice as many operations as either (3.14) or (3.15).

In Section 5, we also need a prolongation operator that may be applied to traces of any polynomial from $\mathcal{P}_{p,\mathbf{x}}$. The space of such traces is denoted as $\mathcal{P}_{p,\mathbf{x}}(\partial\tau_0)$ and contains functions that are continuous on $\partial\tau_0$ and are polynomials of at most degree p on each edge. To describe the lifting for such traces let $v(\mathbf{x}) \in \mathcal{P}_{p,\mathbf{x}}(\partial\tau_0)$, choose any side $T^{(k)}$ and define

$$\Phi_k = v(\mathbf{x}) - F_k(v)(\mathbf{x})|_{\partial\tau_0} .$$

This function is continuous on $\partial\tau_0$, zero on side $T^{(k)}$, and is a polynomial on each of the sides $T^{(k-1)}$ and $T^{(k+1)}$. Therefore, any of the operators defined for the vertex coordinate functions in

(3.14)-(3.16) may be used to lift it. These liftings will generally give different results, and they are not independent of the edge used in the first step. An invariant lifting may be defined as

$$I\!P_\tau v := \frac{1}{3}\sum_{k=1}^{3}[F_k(v)(\mathbf{x}) + \frac{1}{2}(\hat{F}_k^{(k)}(\Phi_k)(\mathbf{x}) + \hat{F}_k^{(k+1)}(\Phi_k)(\mathbf{x}))] \; ; \tag{3.17}$$

however, this requires three times as many operations of the non-invariant lifting. For edge and vertex coordinate functions (3.17) gives the same result as (3.11) and (3.16), respectively.

The basic operation in the prolongations is F_2, which is easily written explicitly [3, 5]. Therefore, the explicit expressions for discrete quasi-harmonic edge and vertex coordinate functions are easily obtained for any type of traces described in Sections 3.1 and 3.2.

Remark 3.1. *In our algorithms, prolongations replace discrete solutions of Laplace's equation with inhomogeneous Dirichlet boundary conditions on the reference element. The former is advantageous (see Section 5 for the operation count) only if the prolongation is less expensive than solving the Dirichlet problem. This is the case for the current state of the art, especially for triangular elements. The situation may be different for hp-methods with square reference elements. In a forthcoming analysis, we will suggest a nearly optimal solver of the Dirchlet problem using p-version finite element approximations on square reference elements.*

4 DD Preconditioners for Discrete Quasi-harmonic Interface Coordinate Functions

4.1 Block-Structured Preconditioners

Let the *interface* be the set $\mathcal{T} = (\bigcup_{r=1}^{\mathcal{R}} \partial\tau_r) \setminus \partial\Omega$, $\mathcal{N}$ be the n umber of unknowns, and $\mathcal{N}_I, \mathcal{N}_{II}, \mathcal{N}_{III}$ be the number of the internal, edge, and vertex unknowns, respectively. The stiffness matrix $\mathbf{K}$ may be written in the 2×2 or 3×3 block forms

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}^{(I)} & \mathbf{K}^{(I,II)} \\ \mathbf{K}^{(II,I)} & \mathbf{K}^{(II)} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{I,II} & \mathbf{K}_{I,III} \\ \mathbf{K}_{II,I} & \mathbf{K}_{II} & \mathbf{K}_{II,III} \\ \mathbf{K}_{III,I} & \mathbf{K}_{III,II} & \mathbf{K}_{III} \end{pmatrix} , \tag{4.1}$$

where $\mathbf{K}^{(I)} = \mathbf{K}_I$ and $\mathbf{K}_L$ are $\mathcal{N}_L \times \mathcal{N}_L$, $L = I, II, III$, sub-matrices corresponding to the internal, side and vertex unknowns, respectively. We have introduced all interface or edge coordinate functions as discrete quasi-harmonic functions by means of explicit prolongation operators. As with square reference elements [15, 23], this makes it possible to effectively use block diagonal preconditioners of the form

$$\mathcal{K} = \mathrm{diag}\left[\mathcal{K}^{(I)},\ \mathcal{K}^{(II)}\right], \qquad \mathcal{K}_v = \mathrm{diag}\left[\mathcal{K}_I,\ \mathcal{K}_{II}, \mathcal{K}_{III}\right] \tag{4.2}$$

where the block diagonal sub-matrices $\mathcal{K}^{(I)}$ and $\mathcal{K}_I$ (which may be identical) and $\mathcal{K}_{II}$ are

$$\mathcal{K}_I = \mathrm{diag}\left[\mathcal{K}_{I,1}, \mathcal{K}_{I,2}, \ldots, \mathcal{K}_{I,\mathcal{R}}\right], \qquad \mathcal{K}_{II} = \mathrm{diag}\left[\mathcal{K}_{II,1}, \mathcal{K}_{II,2}, \ldots, \mathcal{K}_{II,\mathcal{Q}}\right] . \tag{4.3}$$

Here $\mathcal{Q}$ is the number of element edges $T^{(k)}$ within Ω ordered globally. Each block $\boldsymbol{\mathcal{K}}_{I,r}$ has dimension $n_I \times n_I$ with $n_I = p(p-1)/2$ and is related to the internal unknowns on τ_r, $r = 1, 2, \ldots, \mathcal{R}$. Each block $\boldsymbol{\mathcal{K}}_{II,i}$ has dimension $(p-1)\times(p-1)$ and corresponds to the unknowns on the edge in Ω having global number i. If the coefficients of the elliptic equation are constant, then all blocks $\boldsymbol{\mathcal{K}}_{I,r}$, as well as blocks $\boldsymbol{\mathcal{K}}_{II,i}$, may be chosen to be identical on an angular quasi-uniform meshes.

Before considering more efficient preconditioners, we consider the simplest choice when the global stiffness matrix is preconditioned by its block-diagonal part, *i.e.*,

$$\boldsymbol{\mathcal{K}} = \operatorname{diag}\left[\mathbf{K}^{(I)}, \mathbf{K}^{(II)}\right], \qquad \boldsymbol{\mathcal{K}}_v = \operatorname{diag}\left[\mathbf{K}_I, \mathbf{K}_{II}, \mathbf{K}_{III}\right], \tag{4.4}$$

Lemma 4.1. *Suppose that $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{K}}_v$ are defined by (4.4) and*

1. *the triangulation and mappings $\mathcal{X}^{(r)}$ are angular quasiuniform, and*
2. *the edge and vertex coordinate functions of the reference element are discrete quasi-harmonic, i.e., they are obtained from a lifting operator $\mathbb{P}_\tau$ satisfying the inequalities (3.6).*

Then

$$c_1\boldsymbol{\mathcal{K}} \le \mathbf{K} \le 2\boldsymbol{\mathcal{K}}\,. \tag{4.5}$$

If the vertex coordinate functions of the reference element are the linear functions (3.8), then

$$\frac{c_{1,v}}{1+\log p}\boldsymbol{\mathcal{K}}_v \le \mathbf{K} \le 3\boldsymbol{\mathcal{K}}_v\,. \tag{4.6}$$

The constants c_1 and $c_{1,v}$ depend only on θ and μ_1 and μ_2.

Proof. See Section 6.1. □

Remark 4.1. *Lemma 4.1 is easily generalized to the case of many spatial variables. We outline a generalization of (4.5) for a preconditioner with two independent blocks. Let S_h be a quasi-uniform triangulation defined on a polyhedron $\Omega \in \Re^m$, $m \ge 2$, with simplices τ_r having, for simplicity, plane faces. We use similar assumptions on the reference simplex τ_0. In particular,* span $p_{\boldsymbol{\alpha}} = \mathcal{P}_{p,x}$. *If $p_{\boldsymbol{\alpha}}\,|_{\partial\tau_0} \not\equiv 0$, and, thus, $p_{\boldsymbol{\alpha}}$ is a vertex, edge or face coordinate function, then it is discrete quasi-harmonic, i.e., it satisfies inequalities (3.6). Under these assumptions, (4.5) remains valid with the blocks $\mathbf{K}^{(I)}$, $\mathbf{K}^{(II)}$ corresponding to internal and the remaining unknowns, respectively. The proof of these inequalities is the same as with $m = 2$. With $m = 3$, it is also possible to use preconditioners with three and four blocks on diagonal. Inequalities (4.6) hold with a dependence on $\log^k p$ and k adjusted to the properties of the coordinate functions. Suppose, that $m = 3$ and the lower indices I, II, III correspond to internal, face, and the remaining (edge and vertex) unknowns, respectively. Also let $T_{i,0}$ be an edge of τ_0, $i = 1, 2, \ldots, 6$, and the edge and vertex coordinate functions satisfy*

$$\|p_{\boldsymbol{\alpha}}\|_{1/2,\partial\tau_0} \le c\sum_{i=1}^{6}\|p_{\boldsymbol{\alpha}}\|_{0,T_{i,0}}\,.$$

Then, assuming (3.6) to be satisfied, we obtain (4.6) without the $(1+\log p)$ factor.

Once preconditioners $\mathbf{B} = \boldsymbol{\mathcal{K}}$ or $\mathbf{B} = \boldsymbol{\mathcal{K}}_v$ have been defined, they may be used to solve the system (2.7) by the preconditioned conjugate gradient method, GMRES, or another iterative method. The typical operations in a preconditioned iterative method are represented by the simple preconditioned iteration

$$\mathbf{u}^{k+1} = \mathbf{u}^k - \sigma \mathbf{B}^{-1}(\mathbf{K}\mathbf{u}^k - \mathbf{f}) \,. \tag{4.7}$$

To implement (4.7) it is not necessary to know $\mathbf{B}$ or $\mathbf{B}^{-1}$ explicitly. We have to know the sequence of operations necessary to evaluate $\mathbf{B}^{-1}\mathbf{v}$ for any given vector $\mathbf{v}$. As long as preconditioners $\mathbf{B} = \boldsymbol{\mathcal{K}}, \boldsymbol{\mathcal{K}}_v$ are block diagonal, we have to know how to calculate $(\boldsymbol{\mathcal{K}}^{(L)})^{-1}\mathbf{v}^{(L)}$ or $\boldsymbol{\mathcal{K}}_L^{-1}\mathbf{v}_L$, where $\mathbf{v}^{(L)}$ and $\mathbf{v}_L$ are the corresponding restrictions of a vector $\mathbf{v} \in \Re^{\mathcal{N}}$. We may also define $(\boldsymbol{\mathcal{K}}^{(II)})^{-1}\mathbf{v}^{(II)}$ and $\boldsymbol{\mathcal{K}}_{III}^{-1}\mathbf{v}_{III}$ by subsidiary iterative processes with an *a priori* estimated fixed number of iterations. In this case, we call the iteration (4.7) the *leading iteration process.*

4.2 Preconditioning of the Internal Problems

Suppose that $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(0)}$ are the $n \times n$ matrices defined by the identities

$$\mathbf{v}^T\mathbf{A}^{(1)}\mathbf{v} = \int_{\tau_0} \nabla v \cdot \nabla w \, d\mathbf{x} \,, \qquad \mathbf{v}^T\mathbf{A}^{(0)}\mathbf{v} = \int_{\tau_0} vw \, d\mathbf{x}$$

for any polynomials $v, w \in \mathcal{P}_{p,\mathbf{x}}$ and vectors $\mathbf{v}, \mathbf{w} \in \Re^n$ of the coefficients of these polynomials in the basis $(p_{\boldsymbol{\alpha}})$ where $n = (p+1)(p+2)/2$. Let $\mathbf{A}_I^{(1)}$ and $\mathbf{A}_I^{(0)}$ denote the restrictions of $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(0)}$ to the internal degrees of freedom. The blocks $\boldsymbol{\mathcal{K}}_{I,r}$ of the preconditioner $\boldsymbol{\mathcal{K}}_I$ may be defined as either

$$\boldsymbol{\mathcal{K}}_{I,r} = \mu_r \mathbf{A}_I^{(1)} \quad \text{or} \quad \boldsymbol{\mathcal{K}}_{I,r} = \mu_r \mathbf{A}_I^{(1)} + \nu_r h_r^2 \mathbf{A}_I^{(0)} \,, \quad r = 1, 2, \ldots, \mathcal{R} \,. \tag{4.8}$$

The simplest choice for the parameters μ_r and ν_r is unity. Omitting the factor $\nu_r h_r^2 \mathbf{A}_I^{(0)}$ may be justified because $\mathbf{A}_I^{(0)} \le c\mathbf{A}_I^{(1)}$ with an absolute constant c, because $\mathbf{A}_I^{(1)}$ may be much simpler than $\mathbf{A}_I^{(0)}$ and because h may be small. Unless stated otherwise, we use the first of (4.8) with $\mu_r = 1$.

Let (2.2) and the assumptions of Lemma 4.1 on the mappings $\mathcal{X}^{(r)}$ hold. Then $\mathbf{K}_{I,r}$ and $\boldsymbol{\mathcal{K}}_{I,r}$ are spectrally equivalent, *i.e.*,

$$c_{1,I}\boldsymbol{\mathcal{K}}_{I,r} \le \mathbf{K}_{I,r} \le c_{2,I}\boldsymbol{\mathcal{K}}_{I,r}, \qquad r = 1, 2, \ldots, \mathcal{R} \,, \tag{4.9}$$

with the constants depending only on $\alpha^{(1)}, \theta$, μ_1, and μ_2. These inequalities are direct consequences of (2.2) and Lemma 6.1. The use of $\boldsymbol{\mathcal{K}}_{I,r}$, $r = 1, 2, \ldots, \mathcal{R}$, as preconditioners for the internal problems can save computational work even if direct solution procedures are used to solve systems governed by these matrices. The matrices may be factored once for all elements and the leading iterations. Assuming this has been done, the computational work to solve $\mathcal{R}$ internal problems with k_ϵ leading iterations will be $\mathcal{O}(k_\epsilon \mathcal{R} p^4)$ in the worst case of a full $\mathbf{A}_I^{(1)}$.

4.3 Preconditioning of the Interface Problem; Nodal Interface Functions

Suppose $\mathbf{K}_r^{(I)}, \mathbf{K}_r^{(I,II)}, \mathbf{K}_r^{(II,I)}$, and $\mathbf{K}_r^{(II)}$ are the blocks of the element stiffness matrices corresponding to the 2×2 block representation of $\mathbf{K}$ according to (4.1). The sub-matrix $\mathbf{K}^{(II)}$ is the topological sum of the sub-matrices $\mathbf{K}_r^{(II)}$, $r = 1, 2, \ldots, \mathcal{R}$. This is represented by the relationship

$$\mathbf{v}^T \mathbf{K}^{(II)} \mathbf{w} = \sum_r \mathbf{v}_r^T \mathbf{K}_r^{(II)} \mathbf{w}_r \tag{4.10}$$

for all vectors $\mathbf{v}, \mathbf{w} \in \Re^{\mathcal{N}^{(II)}}$, where $\mathcal{N}^{(II)} = \mathcal{N}_I + \mathcal{N}_{II}$, and the vectors $\mathbf{v}_r, \mathbf{w}_r$ are restrictions of $\mathbf{v}, \mathbf{w}$ to nodes on $\partial\tau_r$.

Let us order the $3p$ nodes on $\partial\tau_0$ in counterclockwise order starting from the vertex $V^{(1)}$. By means of the mappings $\mathcal{X}^{(r)}$, this defines the local numbering of the nodes on $\partial\tau_r$, $r = 1, 2, \ldots, \mathcal{R}$. We introduce the $3p \times 3p$ matrices

$$\boldsymbol{\mathcal{D}}_r = \mu_r \boldsymbol{\mathcal{D}} + \nu_r h^3 p^{-1} \mathbf{I}\,, \qquad \boldsymbol{\mathcal{D}} = \begin{pmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & & \cdots & & & \\ & & & \cdots & & \\ & & & -1 & 2 & -1 \\ -1 & & & & -1 & 2 \end{pmatrix}^{1/2}$$

and, for simplicity, assume $\mu_r = \nu_r = 1$. Without changing notation, let us assume that the rows and columns corresponding to the nodes belonging to $\partial\tau_r \bigcap \partial\Omega$ are eliminated from $\boldsymbol{\mathcal{D}}_r$. By assembling the matrices $\boldsymbol{\mathcal{D}}_r$, we may define a matrix $\boldsymbol{\mathcal{D}}^{(II)}$ which satisfies a relationship, similar to (4.10), *i.e.*,

$$\mathbf{v}^T \boldsymbol{\mathcal{D}}^{(II)} \mathbf{w} = \sum_r \mathbf{v}_r^T \boldsymbol{\mathcal{D}}_r \mathbf{w}_r\,. \tag{4.11}$$

In Section 6, we show that $\mathcal{D}^{(II)}$ and $\mathbf{K}^{(II)}$ are spectrally equivalent, but the inversion of $\boldsymbol{\mathcal{D}}^{(II)}$ is still not simple due to its complex structure. Thus, instead of taking this matrix as the preconditioner, we define the inverse $(\boldsymbol{\mathcal{K}}^{(II)})^{-1}$ of the preconditioner by simple iteration with Chebyshev's parameters γ_l:

$$(\boldsymbol{\mathcal{K}}^{(II)})^{-1} = [\mathbf{I} - \sqcap_{l=1}^{n(\epsilon)} (\mathbf{I} - \gamma_l \boldsymbol{\mathcal{D}}^{(II)})](\boldsymbol{\mathcal{D}}^{(II)})^{-1}\,. \tag{4.12}$$

Since we only require that $\boldsymbol{\mathcal{K}}^{(II)}$ and $\mathbf{K}^{(II)}$ be spectrally equivalent, the number $n(\epsilon)$ of iterations may be determined from the condition that the relative error ϵ in the norm induced by matrix $\boldsymbol{\mathcal{D}}^{(II)}$ does not exceed, say, 0.5. In this case, it is easily verified that

$$0.5 \boldsymbol{\mathcal{K}}^{(II)} \leq \boldsymbol{\mathcal{D}}^{(II)} \leq 1.5 \boldsymbol{\mathcal{K}}^{(II)}\,, \tag{4.13}$$

and, consequently, that $\boldsymbol{\mathcal{K}}^{(II)}$ so defined is almost as good a preconditioner for $\mathbf{K}^{(II)}$ as is $\mathcal{D}^{(II)}$. Evidently, the vector $\mathbf{v}^{n(\epsilon)}$ found by the iteration

$$\mathbf{v}^{k+1} = \mathbf{v}^k - \gamma_k (\boldsymbol{\mathcal{D}}^{(II)} \mathbf{v}^k - \boldsymbol{\phi}), \qquad \mathbf{v}^{(0)} = 0, \qquad k = 0, 1, \ldots, n(\epsilon)\,, \tag{4.14}$$

requires at most $\mathcal{O}(p^2)$ operations. Since $\boldsymbol{\mathcal{K}}_{(II,0)}$ does not have such a representation, we first solve (4.20) for $\mathbf{v} = (v_0, v_1, \ldots, v_p)^T$ using $\mathbf{f} = (f_0, f_1, \ldots, f_p)^T$ as a right-hand side. The two components f_0, f_1 of $\mathbf{f}$ are considered as unknown parameters which we determine from the system of two equations that result when $v_0 = 0$ and $v_1 = 0$. Finally, we correct $\mathbf{v}$ so that it becomes the solution of the second of (4.20). The cost of this algorithm is at most $\mathcal{O}(p^2)$ operations. Computations are easily adapted to the situation when $\epsilon_0 = 0$.

A preconditioner $\boldsymbol{\mathcal{K}}_{(II,0)}$, having the same asymptotic cost and effectiveness, is defined in a similar fashion when the traces of the edge coordinate functions are the integrated Legendre polynomials (3.9). One choice is $\boldsymbol{\mathcal{K}}_{(II,0)} = \hat{S}_0$, where $\hat{S}_0$ was defined by Korneev and Jensen [23]. Two factored forms of the corresponding matrix $\boldsymbol{\mathcal{K}}_0^{(II)}$ are given by (3.6) and Remark 3.3 of Korneev and Jensen [23]. From these forms it follows that, once again, the computational cost is at most $\mathcal{O}(p^2)$.

4.6 Vertex Problem, Energy Inequalities for the Three-Block DD Preconditioner

We discuss a three-block preconditioner with independent blocks related to the vertex, edge, and internal unknowns and with the vertex coordinate functions being the linear polynomials (3.8). One choice is $\boldsymbol{\mathcal{K}}_{III} = \mathbf{K}_{III}$. There are several possible simplifications when defining $\boldsymbol{\mathcal{K}}_{III}$ such that the spectra of $\boldsymbol{\mathcal{K}}_{III}$ and $\mathbf{K}_{III}$ are equivalent. We only present one example with $\boldsymbol{\mathcal{K}}_{III} = \bar{\mathbf{K}}_{III}$, where $\bar{\mathbf{K}}_{III}$ is assembled from the artificial element stiffness matrices

$$\bar{\mathbf{K}}_{r,III} = \mu_r \mathbf{A}_{III}^{(1)} \quad \text{or} \quad \bar{\mathbf{K}}_{r,III} = \mu_r \mathbf{A}_{III}^{(1)} + \nu_r h^2 \mathbf{A}_{III}^{(0)} .$$

The matrices $\mathbf{A}_{III}^{(1)}$ and $\mathbf{A}_{III}^{(0)}$ are the blocks of $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(0)}$ corresponding to the vertex unknowns. The values of μ_r and ν_r are either set to unity or chosen to provide a closer approximation of the matrix $\mathbf{K}_{r,III}$.

If the number of elements is large, solving the system

$$\mathbf{L}_{III}\mathbf{u}_{III} = \mathbf{f}_{III} , \qquad \mathbf{L}_{III} = \bar{\mathbf{K}}_{III}, \mathbf{K}_{III} \tag{4.21}$$

may still be difficult. However, these matrices have the typical properties of h-version FEMs and, assuming that (2.2) hold, a variety of iterative methods may be applied, *e.g.*, multigrid, domain decomposition, and fictitious components. Since we use such processes as secondary iterations, it is more efficient to solve (4.21) within reasonable accuracy, provided that the spectral equivalence inequalities

$$0.5\boldsymbol{\mathcal{K}}_{III} \leq \mathbf{K}_{III} \leq 1.5\boldsymbol{\mathcal{K}}_{III} \tag{4.22}$$

hold, where $\boldsymbol{\mathcal{K}}_{III}$ is now defined by the iterative procedure for solving (4.21). If (2.2) hold, the number of iterations sufficient to satisfy (4.22) may be determined from well known *a priori* convergence estimates. If, for example, the multigrid method is used, the number of iterations is independent of h and p and the computational work to solve (4.21) is $\mathcal{O}(h^{-2})$.

A preconditioner $\boldsymbol{\mathcal{K}}_v$ of the form (4.2) and (4.3) with $\boldsymbol{\mathcal{K}}_{L,i}$, $L = I, II$, as defined in previous sections, and $\boldsymbol{\mathcal{K}}_{III}$, as defined here, is less efficient than $\boldsymbol{\mathcal{K}}$. This is due to the formal splitting of the vertex unknowns and the choice of $\boldsymbol{\mathcal{K}}_{II,i}$.

Theorem 4.2. *Let the assumptions of Lemma 4.1 hold, the traces of the edge coordinate functions satisfy (3.10), and the preconditioner $\mathcal{K}_v$ be of the form (4.2) and (4.3). Let us also assume that either $\mathcal{K}_I = \mathbf{K}_I$ or the blocks $\mathcal{K}_{I,r}$ are defined by (4.8), and the blocks $\mathcal{K}_{II}$ and $\mathcal{K}_{III}$ are chosen according to the suggestions of Sections 4.5 and 4.6. Then there exists constants c_1 and c_2 depending only on $\alpha^{(1)}$, θ, μ_1, and μ_2 such that*

$$\frac{c_1}{1+\log^2 p}\mathcal{K}_v \le \mathbf{K} \le c_2(1+\log p)\mathcal{K}_v\,. \tag{4.23}$$

The same inequalities hold if the traces of the edge coordinate functions are given by (3.9) and the matrix $\mathcal{K}_{(II,0)}$ in the blocks of the preconditioner $\mathcal{K}_{II}$ is defined as described in Section 4.5.

Proof. The proof is similar to that of Theorem 5.2 in Korneev and Jensen [23]. □

Remark 4.3. *As with Theorem 4.1, the inequalities (4.23) remain valid if the edge Galerkin coordinate functions are orthogonal to the internal Galerkin coordinate functions for each element. Thus, the two-block preconditioners*

$$\mathcal{K}_T = \mathrm{diag}\,[\mathcal{K}_{II},\, \mathcal{K}_{III}]$$

may be used for solving the system obtained by elimination of the internal unknowns from the rows and columns corresponding to the edge unknowns. The edge coordinate functions may not be dicrete quasi-harmonic functions.

5 Arbitrary Interface Coordinate Functions and DD Preconditioning

In existing computer codes based on the *hp*-version FEM, edge and vertex coordinate functions of a reference element, as a rule, are not discrete quasi-harmonic functions. However, the preconditioners of Section 4 may also be used in this case after adjustment by either transforming the algebraic system or modifying the preconditioners. We discuss both possibilities. In order to distinguish between (2.7) and the algebraic system generated from a basis having non-discrete quasi-harmonic interface functions, we write the latter as

$$\tilde{\mathbf{K}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}}\,. \tag{5.1}$$

The notation $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{K}}_v$ is used for preconditioners that are the counterparts of $\mathcal{K}$ and $\mathcal{K}_v$, and $\tilde{p}_{\boldsymbol{\alpha}}$ for the corresponding reference element coordinate functions. We let $\mathbf{K} = \mathbf{K}_v$ and $\mathbf{f} = \mathbf{f}_v$ when only the edge coordinate functions are discrete harmonic.

5.1 Transformation of the Algebraic System

We transform (5.1) into a system of either the form (2.7) or $\mathbf{K}_v\mathbf{u} = \mathbf{f}_v$ by setting

$$\mathbf{C}\mathbf{u} = \tilde{\mathbf{u}}\,,\quad \mathbf{K} = \mathbf{C}^T\tilde{\mathbf{K}}\mathbf{C}\,,\quad \mathbf{f} = \mathbf{C}^T\tilde{\mathbf{f}}\,,\quad \mathbf{C}_v\mathbf{u} = \tilde{\mathbf{u}}\,,\quad \mathbf{K}_v = \mathbf{C}_v^T\tilde{\mathbf{K}}\mathbf{C}_v\,,\quad \mathbf{f} = \mathbf{C}_v^T\tilde{\mathbf{f}}\,, \tag{5.2}$$

where $\mathbf{C}$ and $\mathbf{C}_v$, respectively, represent transformations of the edge and vertex and edge coordinate functions $\tilde{p}_{\boldsymbol{\alpha}}$ of τ_0 onto the discrete quasi-harmonic coordinate functions

$$p_{\boldsymbol{\alpha}} = I\!P_\tau(\tilde{p}_{\boldsymbol{\alpha}}|_{\partial\tau_0}) \tag{5.3}$$

in the manner of (3.7). Thus, $\mathbf{C}$ are $\mathbf{C}_v$ are defined explicitly through the liftings introduced in Section 3. The transformations (5.2) may be performed on the element level in parallel for each element and without assembling the matrices $\tilde{\mathbf{K}}$, $\mathbf{K}$, or $\mathbf{K}_v$, since the iteration procedure only requires multiplication by $\mathbf{K}$ or $\mathbf{K}_v$. Therefore, if $\tilde{\mathbf{K}}_r$, $\mathbf{K}_r$, and $\tilde{\mathbf{f}}_r$ and $\mathbf{f}_r$ are element stiffness matrices and load vectors corresponding to (5.1), we have

$$\mathbf{K}_r = \mathbf{C}_e^T \tilde{\mathbf{K}}_r \mathbf{C}_e\,, \qquad \mathbf{f}_r = \mathbf{C}_e^T \tilde{\mathbf{f}}_r\,,$$

where $\mathbf{C}_e$ is the restriction of $\mathbf{C}$ to an element. Similar relations are used when only the edge coordinate functions are transformed. Transformations on the element level reduce the computational cost, especially when the coefficients of the elliptic equation are constant on all or some of the elements.

In order to describe the structure of $\mathbf{C}_e$ induced by the liftings of Section 3.3, let $\boldsymbol{\mathcal{M}} = (\boldsymbol{\mathcal{M}}^{(I)}, \boldsymbol{\mathcal{M}}^{(II)})$ and $\tilde{\boldsymbol{\mathcal{M}}} = (\tilde{\boldsymbol{\mathcal{M}}}^{(I)}, \tilde{\boldsymbol{\mathcal{M}}}^{(II)})$ correspond to the rows containing functions from the bases $p_{\boldsymbol{\alpha}}$ and $\tilde{p}_{\boldsymbol{\alpha}}$, where the indices I and II identify the internal and boundary (edge and vertex) coordinate functions, respectively. Assuming that the internal coordinate functions do not change, we write (5.3) in the matrix form $\boldsymbol{\mathcal{M}}^{(II)} = \tilde{\boldsymbol{\mathcal{M}}}(\mathbf{P}_\tau)^T$ with $\mathbf{P}_\tau = (\mathbf{P}_{\tau,I}, \mathbf{I})$, where $\mathbf{I}$ is the $3p \times 3p$ identity matrix and $\mathbf{P}_{\tau,I}$ is a $3p \times p(p-1)/2$ matrix. The matrix $\mathbf{C}_e$ is the same as in the equality $\boldsymbol{\mathcal{M}} = \mathbf{C}_e^T \tilde{\boldsymbol{\mathcal{M}}}$; therefore,

$$\mathbf{C}_e^T = \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{P}_{\tau,I} & \mathbf{I} \end{pmatrix}.$$

In iterative procedures such as (4.7), the matrix $\mathbf{K}$ appears only in the multiplications $\mathbf{v}^k = \mathbf{K}\mathbf{u}^k$. This may be performed element-wise in parallel, *i.e.*, by multiplying $\mathbf{v}_r^k = \mathbf{K}_r \mathbf{u}_r^k$. These,in turn, may be performed by the sequence

$$\mathbf{y}^k = \mathbf{C}_e \mathbf{u}_r^k\,, \qquad \mathbf{z}^k = \tilde{\mathbf{K}}_r \mathbf{y}^k\,, \qquad \mathbf{v}_r^k = \mathbf{C}_e^T \mathbf{z}^k\,. \tag{5.4}$$

The multiplication of a vector by $\tilde{\mathbf{K}}$ would ordinarily be performed; thus, the additional operations needed to calculate $\mathbf{y}^k = \mathbf{C}_e \mathbf{u}_r^k$ and $\mathbf{v}_r^k = \mathbf{C}_e^T \mathbf{z}^k$ are at most $\mathcal{O}(p^3)$ for each element. For $\mathcal{R}$ elements and k_ϵ iterations, this amounts to $\mathcal{O}(\mathcal{R} k_\epsilon p^3)$. Since, by Theorem 4.1, the k_ϵ is independent of p and h, this estimate is asymptotically less than $\mathcal{O}(\mathcal{R}p^4)$, which was obtained for the transformation (5.2) of the global stiffness matrix. The cost of the transformations (5.4) is also less than the cost of multiplications $\mathbf{z}^k = \tilde{\mathbf{K}}_r \mathbf{y}^k$ which, in general, is $\mathcal{O}(\mathcal{R} k_\epsilon p^4)$.

If only the edge coordinate functions are discrete harmonic, the prolongation and transformation matrices, respectively, have the form

$$\mathbf{P}^{(s)} = \begin{pmatrix} \mathbf{P}_{(I)}^{(s)}, & \mathbf{I} \end{pmatrix}, \qquad \mathbf{C}_v = \begin{pmatrix} \mathbf{I} & (\mathbf{P}_{(I)}^{(s)})^T & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{pmatrix}.$$

5.2 Adjustments of Preconditioners

If the edge coordinate functions are not discrete quasi-harmonic, we may introduce preconditioners $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{K}}_v$ for the global stiffness matrix $\tilde{\mathbf{K}}$ that are counterparts of the preconditioners $\mathcal{K}$ and $\mathcal{K}_v$ for $\mathbf{K}$. We define them through their inverses as

$$\tilde{\mathcal{K}}^{-1} = (\mathcal{K}^{(I)})^{+} + \mathbf{P}^T(\mathcal{K}^{(II)})^{-1}\mathbf{P}\,, \quad \tilde{\mathcal{K}}_v^{-1} = \mathcal{K}_I^{+} + (\mathbf{P}^{(s)})^T\mathcal{K}_{II}^{-1}\mathbf{P}^{(s)} + \mathcal{K}_{III}^{+}\,, \tag{5.5}$$

where

$$(\mathcal{K}^{(I)})^{+} = \begin{pmatrix} (\mathcal{K}^{(I)})^{-1}, & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{K}_I^{+} = \begin{pmatrix} \mathcal{K}_I^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{K}_{III}^{+} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{K}_{III}^{-1} \end{pmatrix},$$

and $\mathbf{P} = (\mathbf{P}_I, \mathbf{I})$ is the prolongation operator whose restriction to every element is the matrix $\mathbf{P}_\tau$ introduced in Section 5.1. Again, we need not assemble $\mathbf{P}$ and $\mathbf{P}^{(s)}$ because multiplications by these matrices and their transposes may be done in an element-by-element manner. These preconditioners are justified by the following proposition.

Proposition 5.1. *The inequalities*

$$\nu_1\mathcal{K} \le \mathbf{K} \le \nu_2\mathcal{K}\,, \qquad \sigma_1\mathcal{K}_v \le \mathbf{K}_v \le \sigma_2\mathcal{K}_v\,,$$

respectively, are equivalent to

$$\nu_1\tilde{\mathcal{K}} \le \tilde{\mathbf{K}} \le \nu_2\tilde{\mathcal{K}}\,, \qquad \sigma_1\tilde{\mathcal{K}}_v \le \tilde{\mathbf{K}} \le \sigma_2\tilde{\mathcal{K}}_v\,.$$

Proof. Using the structures of the transformations $\mathbf{C}$ and $\mathbf{C}_v$ and that of $\tilde{\mathcal{K}}^{-1}$ and $\tilde{\mathcal{K}}_v^{-1}$, we verify that (5.5) may be rewritten as

$$\tilde{\mathcal{K}}^{-1} = \mathbf{C}^T\mathcal{K}^{-1}\mathbf{C}, \qquad \tilde{\mathcal{K}}_v^{-1} = \mathbf{C}_v^T\mathcal{K}_v^{-1}\mathbf{C}_v\,.$$

From these relationships, it follows that $\mathcal{K} = \mathbf{C}^T\tilde{\mathcal{K}}\mathbf{C}$ and $\mathcal{K}_v = \mathbf{C}_v^T\tilde{\mathcal{K}}_v\mathbf{C}_v$. Since $\tilde{\mathbf{K}}$, $\mathbf{K}$, and $\tilde{\mathbf{K}}$, $\mathbf{K}_v$ are related by identical transforms, this completes the proof. □

This proposition allows us to reformulate Theorems 4.1 and 4.2 for $\tilde{\mathbf{K}}$ and its preconditioners $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{K}}_v$. Thus, instead of (4.7), we can use the iteration

$$\tilde{\mathbf{u}}^{k+1} = \tilde{\mathbf{u}}^k - \sigma\mathbf{B}^{-1}(\tilde{\mathbf{K}}\tilde{\mathbf{u}}^k - \tilde{\mathbf{f}})\,, \qquad \mathbf{u}^{k+1} = \mathbf{u}^k - \sigma\mathbf{B}^{-1}(\mathbf{C}^T\tilde{\mathbf{K}}\mathbf{C}\mathbf{u}^k - \mathbf{C}^T\mathbf{f})\,, \qquad k = 1, 2, \ldots, \tag{5.6}$$

with the preconditioners $\mathbf{B} = \tilde{\mathcal{K}}, \tilde{\mathcal{K}}_v$. The rate of convergence is the same as for the iteration (4.7) with the preconditioners $\mathcal{K}, \mathcal{K}_v$ (see Theorems 4.1 and 4.2). Comparing the iterations (4.7) and (5.6), we note first that the transformations of the global stiffness matrix required for (4.7), as established, may be more costly than transformations in the manner of (5.4). Additionally, the matrix $\mathbf{K}$ may be denser. The iteration (5.6) appears to have the same asymptotic computational cost. The cost of transformations similar to those used in the second iteration of (5.6) are p-times less expensive than the cost of multiplying by $\tilde{\mathbf{K}}$ or $\mathbf{K}$. This multiplication is one of the two major operations in these iterative procedures.

is exactly

$$\mathbf{v}^{n(\epsilon)} = (\boldsymbol{\mathcal{K}}^{(II)})^{-1}\boldsymbol{\phi} \,. \tag{4.15}$$

Therefore, the product (4.15) may be completed by means of the iteration (4.14).

Let us estimate the computational work assuming that (2.2) are satisfied. It is easy to establish that the condition number of $\boldsymbol{\mathcal{D}}^{(II)}$ is $\mathcal{O}(p/h)$. Therefore, in order to guarantee that $\epsilon \leq 0.5$ and, consequently that (4.13) holds, it is necessary to perform $n(\epsilon) = \mathcal{O}(\sqrt{p/h})$ iterations [32]. At each iteration, the multiplication by $\mathcal{D}^{(II)}$ may be done in parallel for each element.

It follows from (4.11) that

$$\boldsymbol{\mathcal{D}}^{(II)}\mathbf{v} = \sum_r \boldsymbol{\mathcal{D}}_r \mathbf{v}_r \,, \tag{4.16}$$

where $\boldsymbol{\mathcal{D}}_r$ and $\mathbf{v}_r$ are considered to be extended by zero entries. The eigenvalues and eigenfunctions of $\boldsymbol{\mathcal{D}}_r$, $r = 1, 2, \ldots, \mathcal{R}$, are well known and are specified in terms of trigonometric functions. For this reason, each multiplication $\boldsymbol{\mathcal{D}}_r\mathbf{v}_r$ may be performed by the Fast Discrete Fourier Transform (FDFT) in $\mathcal{O}(p \log p)$ operations. The completion of $n(\epsilon)$ multiplications (4.16) requires at most $\mathcal{O}(\mathcal{R}p\sqrt{p/h}\log p)$ operations, which are the main components of the computational cost of the iteration (4.14). In practical implementations, a sufficient number of iterations $n(\epsilon)$ may be either determined from *a priori* estimates, which are easily derived, or numerically as described by Samarskii and Nikolayev [32].

4.4 Spectral Equivalence of the DD Preconditioner and the Global Stiffness Matrix

The use of the suggested preconditioners leads, as we have seen, to a reduction of the computational work when the number of iterations of the leading iteration process (4.7) does not grow rapidly with increasing p or h^{-1}. According to Theorem 4.1, which follows, the number of leading iterations does not depend on p and h. Use of the preconditioner $\boldsymbol{\mathcal{K}}$ does not imply elimination of internal unknowns. However, if this is done, we may use the sub-matrix $\boldsymbol{\mathcal{K}}^{(II)}$ as a spectrally equivalent preconditioner for the system obtained after elimination. The matrix of this system is the Schur complement and we introduce notation

$$\mathbf{K}_{ort}^{(II)} = \mathbf{K}^{(II)} - \mathbf{K}^{(II,I)}(\mathbf{K}^{(I)})^{-1}\mathbf{K}^{(I,II)} \,.$$

Theorem 4.1. *Let the assumptions of Lemma 4.1 hold and the preconditioner $\boldsymbol{\mathcal{K}}$ be of the form (4.2) with $\boldsymbol{\mathcal{K}}^{(I)} = \boldsymbol{\mathcal{K}}_I$. Also, let $\boldsymbol{\mathcal{K}}_I = \mathbf{K}_I$ or the blocks $\boldsymbol{\mathcal{K}}_{I,r}$ be defined as in (4.8), and the sub-matrix $\boldsymbol{\mathcal{K}}^{(II)}$ be defined by (4.12). Then, there exist constants c_1 and c_2, depending only on $\alpha^{(1)}$, θ, μ_1, and μ_2, such that*

$$c_1\boldsymbol{\mathcal{K}} \leq \mathbf{K} \leq c_2\boldsymbol{\mathcal{K}} \,, \qquad c_1\boldsymbol{\mathcal{K}}^{(II)} \leq \mathbf{K}_{ort}^{(II)} \leq c_2\boldsymbol{\mathcal{K}}^{(II)} \,. \tag{4.17}$$

Proof. See the Appendix. □

Remark 4.2. *The matrix $\mathbf{K}_{ort}^{(II)}$ is independent of the continuation of the traces of the vertex and edge functions into τ_0. Thus, the second pair of inequalities (4.17) also hold when the vertex and edge coordinate functions are not discrete quasi-harmonic.*

4.5 Preconditioning of the Edge Problems with Hierarchical Coordinate Functions

Here, we assume that the edge coordinate functions of the reference element are hierarchical, *e.g.*, as given by the expressions (3.9) or (3.10). We would like to specify the blocks $\mathcal{K}_{II,i}$ in the *DD* preconditioner $\mathcal{K}_v$ of the form (4.2) and (4.3), where each block corresponds to one of the sides common to two elements. The main step is to derive the preconditioner for the block of the reference element stiffness matrix corresponding to one of its sides. After proper scaling, we prove that we can use this preconditioner for each of the three blocks on the diagonal of the stiffness matrix of any element that is generated by the edge coordinate functions. We next specify each block $\mathcal{K}_{II,i}$ according to the assembling procedure with the preconditioners for the edge sub-matrices given for each element. The cost of solving the system with the resulting matrix $\mathcal{K}_{II,i}$ is $\mathcal{O}(p^2)$ arithmetical operations.

Let T_i be the common side of triangles τ_{r_1} and τ_{r_2}. In general, we may set

$$\mathcal{K}_{II,i} = \mu_{(i)}\mathcal{K}_{(II,0)} + \nu_{(i)}h^3p^{-1}\mathbf{I}\,, \qquad \mu_{(i)} = \mu_{r_1} + \mu_{r_2}\,, \qquad \nu_{(i)} = \nu_{r_1} + \nu_{r_2}\,, \tag{4.18}$$

with $\mathcal{K}_{(II,0)}$ the same for all edges and $\mu_{(i)} = \nu_{(i)} = 2$ as the simplest choice, which is done below. To specify $\mathcal{K}_{(II,0)}$, Let us introduce the bases

$$\Psi_1 = \begin{pmatrix} 1 \\ s_k \\ (1-s_k^2) \\ (1-s_k^2)s_k \\ \vdots \\ (1-s_k^2)s_k^{p-2} \end{pmatrix}, \ \Psi^{(1)} = \begin{pmatrix} 1 \\ s_k \\ s_k^2 \\ \vdots \\ s_k^p \end{pmatrix}, \ \Psi_2 = \begin{pmatrix} 1 \\ \cos\phi \\ (\cos\phi)^2 \\ \vdots \\ (\cos\phi)^p \end{pmatrix}, \ \Psi^{(2)} = \begin{pmatrix} 1 \\ \cos\phi \\ \cos 2\phi \\ \vdots \\ \cos p\phi \end{pmatrix}.$$

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be the matrices of the transformations $\Psi_1 = \mathbf{C}_1\Psi^{(1)}$ and $\Psi_2 = \mathbf{C}_2\Psi^{(2)}$, and let $\mathbf{\Lambda}$ be the diagonal matrix

$$\mathbf{\Lambda} = \epsilon_0\mathbf{I} + \mathrm{diag}\,[0,\,1,\ldots,p]\,,$$

where ϵ_0 is chosen to satisfy $0 \le \epsilon_0 \le (1+\log p/h)^{-1}$. It is important that $\mathbf{C}_1$ and $\mathbf{C}_2$ be lower-triangular so that solving the systems $\mathbf{C}_k\mathbf{v} = \mathbf{f}$ and $\mathbf{C}_k^T\mathbf{v} = \mathbf{f}$, $k = 1, 2$, requires no more than $\mathcal{O}(p^2)$ operations. Indeed, $\mathbf{C}_1$ is bi-diagonal so that the cost of solving these systems is only $\mathcal{O}(p)$. Deleting the first two rows and columns of

$$\mathcal{K}_0^{(II)} = \mathbf{C}_1\mathbf{C}_2\mathbf{\Lambda}(\mathbf{C}_2)^T(\mathbf{C}_1)^T \tag{4.19}$$

we obtain $\mathcal{K}_{(II,0)}$. Although $\mathcal{K}_0^{(II)}$ is singular when $\epsilon_0 = 0$, $\mathcal{K}_{(II,0)}$ is not. For any ϵ_0 pointed out above, the matrices $\mathcal{K}_{(II,0)}$ satisfying (4.19) are spectrally equivalent uniformly in p and the minor ϵ_0-perturbation in (4.18) for $\mathcal{K}_{II,i}$ may generally be neglected. We use this fact in what follows.

Suppose $\epsilon_0 \ge 0$. From the factored representation (4.19), we may conclude that solving the systems

$$\mathcal{K}_0^{(II)}\mathbf{v} = \mathbf{f}\,, \qquad \mathbf{f} \in \Re^{(p+1)}\,, \qquad \mathcal{K}_{(II,0)}\mathbf{v}_{(II,0)} = \mathbf{f}_{(II,0)}\,, \qquad \mathbf{f}_{(II,0)} \in \Re^{(p-1)}\,, \tag{4.20}$$

6 Appendix

6.1 Proof of Lemma 4.1

Proof. The right sides of (4.5) and (4.6) are Cauchy-Schwarz inequalities. Let us prove the left inequality (4.5) under the additional assumption that $b \equiv 0$. The proof changes slightly if $b \not\equiv 0$. Let us represent $\hat{v}(\mathbf{y}) = v(\mathcal{X}^{(r)}(\mathbf{y}))$, $v \in \mathcal{H}(\tau_r)$, in the form

$$\hat{v} = \hat{v}^{(I)} + \hat{v}^{(II)}, \qquad \hat{v}^{(I)}|_{\partial\tau_0} = 0 \tag{6.1}$$

and use the inequality

$$|\hat{v}^{(I)}|^2_{1,\tau_0} + |\hat{v}^{(II)}|^2_{1,\tau_0} \le 2|\hat{v}|^2_{1,\tau_0} + 3|\hat{v}^{(II)}|^2_{1,\tau_0} \,. \tag{6.2}$$

Since the polynomial $\hat{v}^{(II)}$ is discrete quasi-harmonic, *i.e.*, $\hat{v}^{(II)} = I\!P_\tau \hat{v}^{(II)}|_{\partial\tau_0}$, we can use (3.6) and the trace theorem [1] to write

$$|\hat{v}^{(II)}|^2_{1,\tau_0} \le c_{I\!P}\|\hat{v}^{(II)}\|^2_{1/2,\partial\tau_0} = c_{I\!P}\|\hat{v}\|^2_{1/2,\partial\tau_0} \le cc_{I\!P}\|\hat{v}\|^2_{1,\tau_0} \tag{6.3}$$

with an absolute constant c. Now, by a Bramble-Hilbert Lemma argument, we attain

$$|\hat{v}^{(II)}|_{1,\tau_0} \le c|\hat{v}|_{1,\tau_0} \tag{6.4}$$

with an absolute constant c. Combining (6.2) and (6.4), we have

$$|\hat{v}^{(I)}|^2_{1,\tau_0} + |\hat{v}^{(II)}|^2_{1,\tau_0} \le c|\hat{v}|^2_{1,\tau_0} \,. \tag{6.5}$$

Applying Lemma 6.1 (which follows) we conclude that

$$c_3(\alpha^{(1)}\sin\theta)(|v^{(I)}|^2_{1,\tau_r} + |v^{(II)}|^2_{1,\tau_r}) \le |\hat{v}|^2_{1,\tau_0} \le c_4(\alpha^{(1)}\sin\theta)^{-1}|v|^2_{1,\tau_r} \,, \tag{6.6}$$

where $v = v^{(I)} + v^{(II)}$ is a representation of the polynomial v similar to (6.2). Using (2.2), (2.8), and (6.6), we establish the left inequality (4.5).

We prove the left inequality (4.6) by using Lemma 6.2 (which follows). □

Lemma 6.1. *Let the angular generalized quasiuniformity conditions hold so that (2.3) hold with $h = h(r)$. Then, for every $w \in \mathcal{H}(\tau_0)$, the function $v(\mathcal{X}^{(r)}(\mathbf{y})) = w(\mathbf{y}) \in H^1(\tau_r)$ and for some absolute constants c_1, c_2*

$$c_1(\alpha^{(1)}\sin\theta)|v|^2_{1,\tau_r} \le |w|^2_{1,\tau_0} \le c_2(\alpha^{(1)}\sin\theta)^{-1}|v|^2_{1,\tau_r} \,, \qquad (\sin\theta)(\alpha^{(1)}h)^2 \le d_r \le h^2 \,. \tag{6.7}$$

where $d = d_r(\mathbf{y})$ is the Jacobian of the mapping $\mathcal{X}^{(r)}(\mathbf{y})$.

Proof. This lemma is a particular case of Lemma 7.1 in Korneev [20]. □

Remark 6.1. *In the case of affine mappings, we have*

$$\frac{\rho_{r,1}}{2}|v|_{1,\tau_r} \le \frac{2\sqrt{d_r}}{3} \le \rho_{r,2}|v|_{1,\tau_r} . \tag{6.8}$$

If the mapping is nonlinear, we can introduce a triangle $\hat{\tau}_r$ with straight sides that has common vertices with τ_r. Suppose inequalities (2.3) and (2.4) hold with $n \ge 2$. Then we can consider θ in (6.7) as the smallest angle at the vertices of $\hat{\tau}_r$ and $\alpha^{(1)}$ may be omitted. Similarly, $\rho_{r,1}$, $\rho_{r,2}$ may be taken for $\hat{\tau}_r$. However, the factors $(1-\mathcal{O}(h))$ and $(1+\mathcal{O}(h))$ appear, respectively, on the right and the left sides of (6.7) and (6.8) with the $\mathcal{O}(h)$ term depending on the constant in (2.4).

Lemma 6.2. *Let any polynomial $u \in \mathcal{P}_{p,\mathbf{x}}$ be represented in the form $u = w + v$, where $w \in \operatorname{span}(\mathcal{M}_I \cup \mathcal{M}_{II})$ and $v \in \mathcal{M}_{III}$. Let the bilinear form $a(\cdot,\cdot)$ be*

$$a(\phi,\psi) = \int_{\tau_0} \nabla\phi \cdot \nabla\psi \, d\mathbf{x} \qquad \text{or} \qquad a(\phi,\psi) = \int_{\tau_0} (\nabla\phi \cdot \nabla\psi + \phi\psi) \, d\mathbf{x} .$$

Then, there exists absolute constant c_1 such that

$$\frac{c_1}{1+\log p}[a(w,w)+a(v,v)] \le a(u,u) \le 2[a(w,w)+a(v,v)] .$$

Proof. The proof is exactly the same as that of Lemma 2.2 in Korneev and Jensen [23] for square reference elements, except that the norms and the trace theorems for the triangle τ_0 are used. □

6.2 Proof of Theorem 4.1

Proof. First, we use Lemma 4.1 to conclude that the left pair of inequalities (4.17) can be established by proving

$$c_1\boldsymbol{\mathcal{K}}_I \le \mathbf{K}_I \le c_2\boldsymbol{\mathcal{K}}_I , \qquad c_1\boldsymbol{\mathcal{K}}^{(II)} \le \mathbf{K}^{(II)} \le c_2\boldsymbol{\mathcal{K}}^{(II)} , \tag{6.9}$$

When $\boldsymbol{\mathcal{K}}_I \ne \mathbf{K}_I$, the first pair of inequalities (6.9) follows from (4.9). Thus, it remains to prove the spectral equivalence of $\mathbf{K}^{(II)}$ and $\boldsymbol{\mathcal{K}}^{(II)}$. By (4.13) this is reduced to the inequalities $c_1\boldsymbol{\mathcal{D}}^{(II)} \le \mathbf{K}^{(II)} \le c_2\boldsymbol{\mathcal{D}}^{(II)}$ and, further, due to (4.10) and (4.11) to the inequalities

$$c_1\boldsymbol{\mathcal{D}}_r^{(II)} \le \mathbf{K}_r^{(II)} \le c_2\boldsymbol{\mathcal{D}}_r^{(II)} . \tag{6.10}$$

If the bilinear form $a_\Omega(v,w)$ corresponds to the Laplace operator, these inequalities are satisfied with constants c_1 and c_2 depending on $\alpha^{(1)}$ and θ as a consequence of Corollary 6.1 to Lemma 6.3 (which follows) and Lemma 6.1. For a more general bilinear form, it would also be necessary to use (2.2). In this case, the constants in (6.10) and, consequently, in (6.9) also depend on μ_1 and μ_2. The proof of the inequalities (4.17) involving $\mathbf{K}_{ort}^{(II)}$ follows similar lines. Since Galerkin coordinate functions generating $\mathbf{K}^{(II)}$ and $\mathbf{K}_{ort}^{(II)}$ possess similar properties (some are discrete quasi-harmonic and others are discrete harmonic functions), we establish that

$$c_1\mathbf{K}_{ort}^{(II)} \le \mathbf{K}^{(II)} \le c_2\mathbf{K}_{ort}^{(II)} .$$

Using this with (6.9) completes the proof. □

Let $\boldsymbol{\zeta}^{\boldsymbol{\alpha}}$ be the notation for the nodes on the boundary $\partial\tau_0$ described in Section 3.1, where $\boldsymbol{\alpha} = (k, l, 0)$, $k = 1, 2, 3$, $l = 1, 2, \ldots, p-1$. Let $\boldsymbol{\eta}^{\boldsymbol{\alpha}}$ with the same indices $\boldsymbol{\alpha}$ represent nodes distributed uniformly on $\partial\tau_0$ such that each edge is divided into p equal parts. The positions of nodes $\boldsymbol{\zeta}^{\boldsymbol{\alpha}} \in T^{(k)}$ are specified by their coordinates $s_k = s_k^{(l)}$ given by (3.2). By $\mathcal{L}_1(\partial\tau_0)$, we denote the space of continuous functions on $\partial\tau_0$ that are linear between the nodes $\boldsymbol{\eta}^{\boldsymbol{\alpha}}$.

Lemma 6.3. *For any $v \in \mathcal{P}_{p,\mathbf{x}}(\partial\tau_0)$ and $w \in \mathcal{L}_1(\partial\tau_0)$ such that*

$$v(\boldsymbol{\zeta}^{\boldsymbol{\alpha}}) = w(\boldsymbol{\eta}^{\boldsymbol{\alpha}}) , \qquad \forall\, \boldsymbol{\eta}^{\boldsymbol{\alpha}} \in \partial\tau_0 ,$$

the norms $\|\cdot\|_{1/2,\partial\tau_0}$ and quasinorms $|\cdot|_{1/2,\partial\tau_0}$ are equivalent, i.e.,

$$\|w\|_{1/2,\partial\tau_0} \asymp \|v\|_{1/2,\partial\tau_0} , \qquad |w|_{1/2,\partial\tau_0} \asymp |v|_{1/2,\partial\tau_0} . \tag{6.11}$$

Proof. Similar inequalities were established by Korneev and Jensen ([23], Lemma 6.1) for square reference elements with a tensor-product polynomial space. Although the geometric form and the space of the element under consideration are different, the proof follows the same lines. □

The stiffness matrix of the reference element may be represented in a block form as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^{(I)} & \mathbf{A}^{(I,II)} \\ \mathbf{A}^{(II,I)} & \mathbf{A}^{(II)} \end{pmatrix}, \qquad \mathbf{A}^{(II)} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{II,III} \\ \mathbf{A}_{III,II} & \mathbf{A}_{III} \end{pmatrix} ,$$

and factored as the product

$$\mathbf{A} = \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{A}^{(II,I)}(\mathbf{A}^{(I)})^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A}^{(I)} & 0 \\ 0 & \hat{\mathbf{S}}^{(II)} \end{pmatrix} \begin{pmatrix} \mathbf{I} & (\mathbf{A}^{(I)})^{-1}\mathbf{A}^{(I,II)} \\ 0 & \mathbf{I} \end{pmatrix},$$

$$\hat{\mathbf{S}}^{(II)} = \mathbf{A}^{(II)} - \mathbf{A}^{(II,I)}(\mathbf{A}^{(I)})^{-1}\mathbf{A}^{(I,II)} .$$

Corollary 6.1. *Suppose $\mathbf{A} = \mathbf{A}^{(1)}$ and the edge and vertex coordinate functions are discrete quasi-harmonic polynomials as described in Section 3.1. Then*

$$\boldsymbol{\mathcal{D}} \asymp \hat{\mathbf{S}}^{(II)} \asymp \mathbf{A}^{(II)} . \tag{6.12}$$

Proof. For each edge, we define the mapping $\bar{T}^{(k)} \to \bar{T}^{(k)}$ in the form $s_k = -\cos\frac{\pi}{2}(t_k + 1)$, $s_k, t_k \in [-1, 1]$. These mappings define the transformation $\partial\tau_0 \to \partial\tau_0$, which is denoted as $\mathbf{x} = \mathcal{Y}(\mathbf{y})$. It is continuous on $\partial\tau_0$ and preserves the norm $\|\cdot\|_{1/2,\partial\tau_0}$. The latter is proved by a manner similar to (6.10) of Korneev and Jensen [23]. Thus, if $v(\mathbf{x}) \in \mathcal{P}_{p,\mathbf{x}}(\partial\tau_0)$ and $v_{\cos}(\mathbf{y}) = v(\mathcal{Y}(\mathbf{y}))$, then

$$\|v\|_{1/2,\partial\tau_0} \asymp \|v_{\cos}\|_{1/2,\partial\tau_0} . \tag{6.13}$$

Equivalence in the spectrum of the matrices $\mathcal{D}$ and $S^{(II)}$ follows from (6.13) and

$$|v|_{1/2,\partial\tau_0} \asymp (\mathbf{v}^{(II)})^T \hat{\mathbf{S}}^{(II)} \mathbf{v}^{(II)} , \qquad |w|_{1/2,\partial\tau_0} \asymp (\mathbf{v}^{(II)})^T \boldsymbol{\mathcal{D}} \mathbf{v}^{(II)} . \tag{6.14}$$

The left relationship (6.14) simply expresses the fact that the equivalent definition of the quasi-norm $|\cdot|_{1/2,\partial\tau_0}$ is

$$|v|_{1/2,\partial\tau} = \inf |\phi|_{1/2,\partial\tau} , \tag{6.15}$$

where the infimum is taken over all $\phi \in H^1(\tau_0)$ satisfying the boundary condition $\phi(\mathbf{x}) = v(\mathbf{x})$, $\mathbf{x} \in \partial\tau_0$. The right relationship (6.14) is known from the studies of domain decomposition techniques for the h-version FEM, see, *e.g.*, Dryja [11]. It is obtained by interpolation between the spaces $H^1(\partial\tau_0) \cap \mathcal{L}_1(\partial\tau_0)$ and $L_2(\partial\tau_0) \cap \mathcal{L}_1(\partial\tau_0)$.

The spectral equivalence of $\hat{\mathbf{S}}^{(II)}$ and $\mathbf{A}^{(II)}$ follows from (6.15), the trace theorem, and the definition of the interface coordinate functions as discrete quasi-harmonic polynomials, *i.e.*, from (3.6), $\forall\, v \in \mathcal{P}^{(II)}_{p,\mathbf{x}} = \mathrm{span}(\mathcal{M}^{(II)})$. □

6.3 Notation

The notation used throughout this work follows.

- $\mathcal{P}_{p,\mathbf{x}}$ is the space of polynomials of *total degree* not exceeding $p \geq 1$; $\mathcal{P}_{p,\mathbf{x}}(\partial\tau_0)$ is the space of traces of these polynomials on $\partial\tau_0$.
- $(\cdot,\cdot)_\Omega$, $\|\cdot\|_\Omega = \|\cdot\|_{0,\Omega}$ are the scalar product and the norm in $L^2(\Omega)$.
- $|\cdot|_{k,\Omega}$, $\|\cdot\|_{k,\Omega}$ are the semi-norm and the norm in the Hilbert space $H^k(\Omega)$, *i.e.*,

$$|v|^2_{k,\Omega} = \sum_{|q|=k} \int_\Omega (D^q_x v)^2 dx \;, \quad \|v\|^2_{k,\Omega} = \|v\|^2_{0,\Omega} + \sum_{l=1}^{k} |v|^2_{l,\Omega} \;,$$

$$D^q_x v := \partial^{|q|} v / \partial x_1^{q_1} \partial x_2^{q_2} \;, \quad q = (q_1, q_2) \;, \quad q_1, q_2 \geq 0 \;, \quad |q| = q_1 + q_2 \;.$$

- $\overset{o}{H}{}^1(\Omega)$ is the subspace of functions from $H^1(\Omega)$ having zero traces on $\partial\Omega$.
- $\|\cdot\|_{1/2,I}$, ${}_0\|\cdot\|_{1/2,I}$ with $I = (a,b)$ are norms in the space $H^{1/2}(I)$ and the subspace ${}_0H^{1/2}(I) \subset H^{1/2}(I)$ of functions having zero values at $x = a, b$ [1]. These norms for $I_* = (a,b)$ are defined as

$$\|v\|^2_{1/2,I} := \|v\|^2_{0,I} + |v|^2_{1/2,I} \;, \quad |v|^2_{1/2,I_*} := \int_a^b \int_a^b \left(\frac{v(x) - v(y)}{x - y} \right)^2 dxdy \;,$$

$${}_0\|v\|^2_{1/2,I_*} := \|v\|^2_{1/2,I_*} + 2\int_a^b \frac{v^2(x)}{x-a} dx + 2\int_a^b \frac{v^2(x)}{b-x} dx \;.$$

- The norm $\|\cdot\|_{1/2,T^{(k)}}$, where $T^{(k)}$ is an edge of τ_0 joining the vertices $V^{(k-1)}$ and $V^{(k)}$ is defined analogously to $\|\cdot\|_{1/2,I}$. For example, for the edge $T^{(2)}$ on the line $x_2 = 0$, we have

$$\|v\|^2_{1/2,T^{(2)}} := \|v\|^2_{0,T^{(2)}} + |v|^2_{1/2,T^{(2)}} \;, \quad |v|^2_{1/2,T^{(2)}} := \int_{-1}^{1} \int_{-1}^{1} \left(\frac{v(t,0) - v(\tau,0)}{t - \tau} \right)^2 dtd\tau \;.$$

- We also need the norm

$$\|v\|^2_{1/2,\partial\tau_0} = \sum_{k=1}^{3} \|v\|^2_{1/2,T^{(k)}} + \sum_{k=1}^{3} \int_0^1 \frac{(v_{-,k}(t) - v_{+,k}(t))^2}{|t|} dt,$$

where $v_{-,k}$ and $v_{+,k}$ denote the values of v on the sides $T^{(k)}$ and $T^{(k+1)}$, respectively, at a distance t from the vertex $V^{(k)}$. We set $|v|^2_{1/2,\partial\tau_0} := \|v\|^2_{1/2,\partial\tau_0} - \|v\|^2_{0,\partial\tau_0}$. The norm and seminorm defined in this way for the space $H_2^{1/2}(\partial\tau_0)$ are equivalent to $\|v\|_{1/2,\partial\tau_0} := \inf \|w\|_{1,\tau_0}$ and $|v|_{1/2,\partial\tau_0} := \inf |w|_{1,\tau_0}$ with infimae taken over $w \in H^1(\tau_0)$ for which $w = v$ on $\partial\tau_0$.

- $C^{(l)}$ is the class of boundaries $\partial\Omega$ composed of a finite number of l-times continuously differentiable pieces which meet at angles distinct from 0 and π.

- $\mathcal{A}^+$ is the pseudo-inverse of $\mathcal{A}$.

- The symbols $\prec$, $\succ$, $\asymp$ denote one sided and two sided inequalities, which hold with some absolute constants omitted.

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