A Mixed-Penalty Biphasic Finite Element Formulation Incorporating Viscous Fluids and Material Interfaces

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Abstract - The fluid viscosity term of the fluid phase constitutive equation and the interface boundary conditions between biphasic, solid and fluid domains have been incorporated into a mixed-penalty finite element formulation of the linear biphasic theory for hydrated soft tissue. The finite element code can now model a single-phase viscous incompressible fluid, or a single-phase elastic solid, as limiting cases of a biphasic material. Interface boundary conditions allow the solution of problems involving combinations of biphasic, fluid and solid regions. To incorporate these conditions, the volume-weighted mixture velocity is introduced as a degree of freedom at interface nodes, so that the kinematic continuity conditions are satisfied by conventional finite element assembly techniques. Results comparing our numerical method with an independent, analytic solution for the problem of Couette flow over rigid and deformable porous biphasic layers show that the finite element code accurately predicts the viscous fluid flows and deformation in the porous biphasic region. Thus, the analysis can be used to model the interface between synovial fluid and articular cartilage in diarthrodial joints. This is an important step toward modeling and understanding the mechanisms of joint lubrication and another step toward fully modeling the *in vivo* behavior of a diarthrodial joint.

Keywords -- biphasic finite element formulation, fluid viscosity, material interface, Couette flow

INTRODUCTION

The biomechanics of diarthrodial joints has been and continues to be of great interest to researchers. It is believed that a better comprehension of the friction, lubrication and wear of diarthrodial joints will lead to a better understanding of the pathogenesis of osteoarthritis [7, 15, 18]. The two materials common to all diarthrodial joints are articular cartilage (a porous, permeable, viscoelastic, multi-phase material) and synovial fluid (a non-Newtonian fluid) [17]. Thus, changes in the mechanical properties of articular cartilage or the rheological properties of synovial fluid may lead to increase friction and wear within the joints, leading to degenerative joint disorders, such as osteoarthritis [1].

Since interstitial fluid flow plays an important role in the deformational behavior of articular cartilage, a realistic model of cartilage must incorporate the fluid component as a distinct phase of the system within the tissue. Mow and co-workers [17] have developed a biphasic continuum model that can accurately represent the deformational behavior of hydrated soft tissue such as articular cartilage or meniscus in the human knee joint. In order to solve the governing biphasic equations for realistic joint mechanics problems with complex geometry and boundary conditions, numerical methods, such as the finite element method, must be used. Various finite element formulations of the biphasic theory have been developed corresponding to the linear biphasic theory [19, 20, 25], the nonlinear biphasic theory including strain-dependent permeability [21] and finite deformation theories [22, 23]. However, these formulations consider only the biphasic soft tissue and do not include the dynamic interaction between the articular cartilage and its surrounding fluid.

Understanding the modes of diarthrodial joint lubrication requires an analysis incorporating both the soft tissues and the synovial fluid, including proper formulation of the boundary conditions at the interface between articular cartilage and synovial fluid. In the context of the biphasic theory, Hou *et al.* [10] have derived the boundary conditions at the interface between a viscous fluid and a biphasic material. In Hou's study, single-phase continua such as solid or fluid

are modeled as a mixture with only one phase. Thus the interfacial conditions derived for the mixture theory are also applicable to the boundary between mixtures and single-phase continua. These boundary conditions are derived from balances of mass, momentum and energy and must be consistent with the imposed kinematic boundary conditions. In addition, the imposed kinematic boundary conditions must reduce to the kinematic conditions in single phase continuum mechanics when mixtures reduce to single phase continua.

While biphasic contact finite element formulations [6] and analytic contact solutions [3, 4] do exist, these do not account for fluid viscosity. Few studies have used numerical investigations to understand the importance of synovial fluid in joint lubrication, for example [8, 9, 12, 24]. In the present study, both fluid viscosity and the boundary conditions at the cartilage-synovial fluid interface [10] will be incorporated into the mixed-penalty finite element formulation [19] of the linear biphasic theory. This will be done by introducing a volume-weighted mixture velocity as a degree of freedom at the interface nodes. With proper treatment of the constraints and the assembly of elements, the interface kinematic boundary conditions are satisfied exactly in the finite element solution. We will then investigate problems that involve solid, fluid, and biphasic regions.

LINEAR BIPHASIC THEORY WITH VISCOUS FLUID

In this study, viscosity is added to the fluid phase of the linear biphasic theory so that single phase continua can be treated as limiting cases in which the solid or fluid content is zero. The subchondral bone will be modeled as an elastic solid and the synovial fluid as a viscous fluid. In the linear biphasic model, the phases are assumed to be immiscible, incompressible and to undergo small deformations. The governing differential equations of the linear biphasic model with a viscous, incompressible fluid for a material domain Ω having volume V with boundary Γ are (in tensor form) [17]:

$$\nabla \cdot \left(\phi^s \mathbf{v}^s + \phi^f \mathbf{v}^f \right) = 0 \tag{1}$$

$$\nabla \cdot \boldsymbol{\sigma}^\alpha + \boldsymbol{\Pi}^\alpha = \mathbf{0}, \quad \alpha = s, f \tag{2}$$

$$\boldsymbol{\sigma}^s = -\phi^s p\mathbf{I} + \mathbf{C}^s\boldsymbol{\varepsilon}^s \tag{3}$$

$$\boldsymbol{\sigma}^f = -\phi^f p\mathbf{I} + \mathbf{C}^f\dot{\boldsymbol{\varepsilon}}^f \tag{4}$$

$$\boldsymbol{\Pi}^s = -\boldsymbol{\Pi}^f = K\left(\mathbf{v}^f - \mathbf{v}^s\right). \tag{5}$$

They correspond to the continuity equation for the mixture (1), momentum equations for each phase (2), constitutive equations (3),(4) and the diffusive drag force (5). The superscripts 'f' and 's' denote quantities associated with the fluid and solid phases respectively, ϕ^s and ϕ^f are volume fractions which sum to unity, $\mathbf{v}$ is the velocity vector, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\mathbf{C}^\alpha$ are the fourth-rank tensors of material coefficients, $\boldsymbol{\Pi}$ is the diffusive momentum exchange between the two phases, p is the apparent pressure, $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, $\dot{\boldsymbol{\varepsilon}}$ is the rate of deformation tensor and K is the diffusive drag coefficient [14]:

$$K = \frac{\phi^{f^2}}{\kappa}, \tag{6}$$

with κ the tissue permeability constant. Note that $\mathbf{C}^s$ has only two independent terms for the linear, isotropic solid phase, and $\mathbf{C}^f$ will have only the fluid viscosity term, μ^f, multiplying the rate of deformation, for the viscous, incompressible, Newtonian fluid considered here. Boundary conditions on solid displacements, fluid velocities, solid traction and fluid traction are required on the surfaces Γ^{u^s}, Γ^{u^f}, Γ^{σ^s} and Γ^{σ^f}, respectively. To complete the problem definition the appropriate initial conditions must also be specified; these will be specified in the context of each example problem. From the above equations the finite element formulation of the linear biphasic theory with a viscous fluid can now be developed.

FINITE ELEMENT FORMULATION

Mixed-Penalty Formulation

The mixed-penalty formulation of Spilker and Maxian [16, 19] for the linear biphasic theory is extended here to allow for viscosity in the fluid phase of the biphasic material. Most details of

the formulation are omitted here since they are similar to the derivation without viscosity [5, 16, 19]. In the mixed-penalty formulation, the continuity equation is replaced by the penalty form,

$$\nabla \cdot \left(\phi^s \mathbf{v}^s + \phi^f \mathbf{v}^f \right) + \frac{p}{\beta} = 0 , \tag{7}$$

where β is a user-defined penalty number. In the Galerkin weighted residual method, this penalty form of the continuity equation, the momentum equations and natural boundary conditions are then each multiplied by an appropriate weighting function, integrated over the domain, summed together and equated to zero

$$\begin{aligned} &\int_\Omega w^c \left[\nabla \cdot \left(\phi^s \mathbf{v}^s + \phi^f \mathbf{v}^f \right) + \frac{p}{\beta} \right] d\Omega + \\ &\int_\Omega \mathbf{w}^s \left(\nabla \cdot \boldsymbol{\sigma}^s + \boldsymbol{\Pi}^s \right) d\Omega + \int_\Omega \mathbf{w}^f \left(\nabla \cdot \boldsymbol{\sigma}^f + \boldsymbol{\Pi}^f \right) d\Omega + . \\ &\int_{\Gamma^{\sigma^s}} \mathbf{h}^s \left(\boldsymbol{\sigma}^s \mathbf{n} - \bar{\mathbf{t}}^s \right) d\Gamma + \int_{\Gamma^{\sigma^f}} \mathbf{h}^f \left(\boldsymbol{\sigma}^f \mathbf{n} - \bar{\mathbf{t}}^f \right) d\Gamma = 0 \end{aligned} \tag{8}$$

Equations (3) through (6) are substituted into the weighted residual, Eq. (8), and the divergence theorem is applied to the terms containing derivatives of stress. By proper manipulation the weak form of the weighted residual is obtained, after which the finite element discretization can be applied. This weak form contains solid displacement, solid velocity, fluid velocity and pressure as independent field variables. To guarantee that all the integrals in the weak form remain finite, the solid and fluid velocities/displacements are assumed to be C^0 continuous, while the pressure is allowed to be discontinuous between elements.

The field variables are interpolated within the element in terms of a corresponding set of nodal values in the following form:

$$\mathbf{u}^s = \mathbf{N}^s \mathbf{d}^{es}, \quad \mathbf{v}^s = \mathbf{N}^s \mathbf{v}^{es}, \quad \mathbf{v}^f = \mathbf{N}^f \mathbf{v}^{ef}, \quad p = \mathbf{N}_p \mathbf{p}^e , \tag{9}$$

where $\mathbf{d}^{es}$, $\mathbf{v}^{es}$ and $\mathbf{v}^{ef}$ are the nodal solid displacements, nodal solid velocities and nodal fluid velocities for a typical element ‘e’ and $\mathbf{p}^e$ is the vector of coefficients of the the pressure

interpolation for an element. In general, the interpolation functions used for the solid and fluid phases are the same, so we set $\mathbf{N}^s=\mathbf{N}^f=\mathbf{N}$. Since this is a Galerkin approximation, the interpolation used for the weighting functions are the same as those used for the field variables, with $\mathbf{w}^{es}$, $\mathbf{w}^{ef}$ and $\mathbf{w}^{ec}$ as arbitrary coefficients for element 'e'. For a finite element program, the stress and strain tensors are represented as vectors, and the corresponding fourth-rank tensors of material coefficients are represented as square matrices. In this reduced index form, the matrix $\mathbf{B}$ will represent the symmetric gradient operation on the spatial interpolation, $\mathbf{N}$, and the vector $\mathbf{m}$ will be the representation of the Kronecker delta.

The element interpolations are substituted into the weak form of the weighted residual, producing the following set of matrix equations:

$$\sum_e \begin{Bmatrix} \mathbf{w}^{es^T} \\ \mathbf{w}^{ef^T} \\ \mathbf{w}^{ec^T} \end{Bmatrix} \begin{bmatrix} \mathbf{c} & -\mathbf{c} & \mathbf{0} \\ -\mathbf{c} & \mathbf{c}+\mathbf{k}^{ef} & \mathbf{0} \\ -\phi^s\mathbf{a}^T & -\phi^f\mathbf{a}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{v}^{es} \\ \mathbf{v}^{ef} \\ \mathbf{0} \end{Bmatrix} + \sum_e \begin{Bmatrix} \mathbf{w}^{es^T} \\ \mathbf{w}^{ef^T} \\ \mathbf{w}^{ec^T} \end{Bmatrix} \begin{bmatrix} \mathbf{k}^{es} & \mathbf{0} & -\phi^s\mathbf{a} \\ \mathbf{0} & \mathbf{0} & -\phi^f\mathbf{a} \\ \mathbf{0} & \mathbf{0} & \frac{1}{\beta}\mathbf{k}_p \end{bmatrix} \begin{Bmatrix} \mathbf{d}^{es} \\ \mathbf{0} \\ \mathbf{p}^e \end{Bmatrix} = \sum_e \begin{Bmatrix} \mathbf{w}^{es^T} \\ \mathbf{w}^{ef^T} \\ \mathbf{w}^{ec^T} \end{Bmatrix} \begin{Bmatrix} \mathbf{f}^{es} \\ \mathbf{f}^{ef} \\ \mathbf{0} \end{Bmatrix}, \tag{10}$$

where the element submatrices are defined as:

$$\mathbf{a} = \int_{\Omega_e} (\mathbf{mB})^T \mathbf{N}_p d\Omega \tag{11}$$

$$\mathbf{k}^{e\alpha} = \int_{\Omega_e} \mathbf{B}^T \mathbf{C}^{\alpha} \mathbf{B} d\Omega, \quad \alpha = s, f \tag{12}$$

$$\mathbf{c} = \int_{\Omega_e} K\mathbf{N}^T\mathbf{N} d\Omega \tag{13}$$

$$\mathbf{k}_p = \int_{\Omega_e} \mathbf{N}_p^T\mathbf{N}_p d\Omega, \tag{14}$$

and all other terms are as defined before. The current formulation differs from the original mixed-penalty formulation [16] by the addition of the fluid stiffness term, $\mathbf{k}^{ef}$.

Because pressure is discontinuous between elements, the pressure coefficients, $\mathbf{p}^e$, can be

eliminated at the element level by using the last equation in matrix equation (10). This produces the final set of matrix equations for the mixed-penalty formulation:

$$\sum_e \begin{Bmatrix} \mathbf{w}^{es^T} \\ \mathbf{w}^{ef^T} \end{Bmatrix} \left\{ \beta \begin{bmatrix} \phi^{s^2}\mathbf{A} & \phi^s\phi^f\mathbf{A} \\ \phi^s\phi^f\mathbf{A} & \phi^{f^2}\mathbf{A} \end{bmatrix} + \begin{bmatrix} \mathbf{c} & -\mathbf{c} \\ -\mathbf{c} & \mathbf{c}+\mathbf{k}^{ef} \end{bmatrix} \right\} \begin{Bmatrix} \mathbf{v}^{es} \\ \mathbf{v}^{ef} \end{Bmatrix} + \sum_e \begin{Bmatrix} \mathbf{w}^{es^T} \\ \mathbf{w}^{ef^T} \end{Bmatrix} \begin{bmatrix} \mathbf{k}^{es} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{d}^{es} \\ \mathbf{0} \end{Bmatrix} = \sum_e \begin{Bmatrix} \mathbf{w}^{es^T} \\ \mathbf{w}^{ef^T} \end{Bmatrix} \begin{Bmatrix} \mathbf{f}^{es} \\ \mathbf{f}^{ef} \end{Bmatrix}, \tag{15}$$

where $\mathbf{A}=\mathbf{a}\mathbf{k}_p^{-1}\mathbf{a}^T$.

The element matrices are then assembled into their respective global matrices by the standard Boolean assembly operator. Then, for arbitrary weighting functions, the assembled coefficients of the weighting functions must be nonzero and the mixed-penalty formulation of the linear biphasic theory with viscous fluid produces the following system of first order differential equations,

$$\mathbf{C}\mathbf{v}+\mathbf{K}\mathbf{d}=\mathbf{F}, \tag{16}$$

where **C** includes the penalty terms, diffusive drag and fluid viscosity, and **K** is the assembled counterpart of the block two-by-two element stiffness matrix. The vectors **d** and **v** are the assembled degree of freedom vectors and **F** is the assembled force vector. At the global level, the ordering of the degrees of freedom in **d** and **v** are different from the ordering at the element level. For computational efficiency when performing element integrals, the degrees of freedom at the element level are ordered as all solid degrees of freedom followed by all fluid degrees of freedom. At the global level, the degrees of freedom are ordered alternately as solid and fluid degrees of freedom for each node.

Solution of the System of Equations

The system of equations in Eq. (16) is solved using finite difference techniques. Given times t_{m+1} and t_m separated by a time increment $\Delta t=t_{m+1}-t_m$, the system of equations at time t_{m+1} is solved using the generalized trapezoidal family of first order finite difference rules [11].

$$\begin{aligned} \mathbf{d}_{m+1} &= \mathbf{d}_m + \Delta t \mathbf{v}_{m+\omega} \\ \mathbf{v}_{m+\omega} &= (1-\omega)\mathbf{v}_m + \omega \mathbf{v}_{m+1}, \quad \omega \in [0,1] \end{aligned} \tag{17}$$

Writing Eq. (16) at $t=t_{m+1}$ and substituting from Eq. (17) yields the following set of matrix equations to be solved at each time step:

$$(\mathbf{C} + \omega \Delta t \mathbf{K})\mathbf{v}_{m+1} = \mathbf{F}_{m+1} - \mathbf{K}\left(\mathbf{d}_m + (1-\omega)\Delta t \mathbf{v}_m\right). \tag{18}$$

Once a set of initial conditions is given, this set of equations can be solved recursively. In our study, an unconditionally stable implicit rule ($\omega \geq 1/2$) is used [21].

Selection of Penalty Number

The penalty term, β, is a user-defined number that should be large enough to enforce constraint conditions yet not so large that the solution of the governing equations become ill-conditioned. Through earlier investigations on penalty formulations of the linear biphasic theory[21], and extensions for a viscous fluid phase, we choose the penalty term according to the following guideline :

$$\beta = c t_0 \frac{\lambda^s + 2\mu^s}{\phi^s} + c\frac{\mu^f}{\phi^s}, \tag{19}$$

where t_0 is a reference time (*e.g.* ramp time), c is a computer-dependent parameter related to the accuracy of the numerical calculations (c is usually chosen in the range 10^7 to 10^9), λ and μ are the solid phase Lamé parameters and μ^f is the viscosity of the fluid. Recognize that for parameters representative of articular cartilage, the second term above can be neglected. It has been observed that variations in β by ± 2 orders of magnitude have negligible affect on the solution.

In the limiting case of a single phase fluid, however, the penalty number is chosen according to the criteria established for penalty solutions of viscous incompressible flows [11]:

$$\beta = c\mu^f, \tag{20}$$

where μ^f and c are as described above.

FINITE ELEMENT FORMULATION OF THE INTERFACE BOUNDARY CONDITIONS

The development of the interface boundary conditions can be found in Hou *et al.* [10]. Since both the solid and fluid phases for the biphasic mixture are intrinsically incompressible, the jump condition for the mass across a surface of discontinuity is

$$\left[\left[\phi^s \mathbf{v}^s + \phi^f \mathbf{v}^f\right]\right] \cdot \mathbf{n} = 0, \tag{21}$$

where [[()]] represents the jump in quantity (), **n** is the normal to the surface of discontinuity and the argument here is the volume-weighted mixture velocity. In order to account for the viscous interactions at the interface, an additional kinematic boundary condition has to be imposed. For viscous fluids, the "no-slip" condition at the fluid/solid interface must be specified. Under ordinary circumstances, viscous fluids adhere to solid (non-porous) surfaces, such that

$$\lim_{\mathbf{x}\to\mathbf{x}_s} \mathbf{v}^f(\mathbf{x}) = \mathbf{v}_{surface}(\mathbf{x}_s), \tag{22}$$

where $\mathbf{x}_s$ is the coordinate for the surface and $\mathbf{v}_{surface}$ is the velocity of the surface. Since a single-phase material is treated as a special case of a biphasic medium, it is proposed that all kinematic boundary conditions at a surface of discontinuity within a biphasic material reduce, in the limiting case of $\phi^f \to 0$ on one side of the surface and $\phi^s \to 0$ on the other side, to the condition described in Eq. (22).

The following jump condition satisfies the above condition:

$$\left[\left[\tilde{\phi}^s \mathbf{v}^s + \tilde{\phi}^f \mathbf{v}^f\right]\right] \cdot \boldsymbol{\tau} = 0, \tag{23}$$

where $\tilde{\phi}^s$ and $\tilde{\phi}^f$ are the surface area fraction of the solid and fluid phases respectively, and $\boldsymbol{\tau}$ is any tangent vector of the surface. Since the pore structure of articular cartilage is assumed to be isotropic, the area fractions can be shown to be equivalent to the volume fractions. The jump condition in Eq. (23) is called a "pseudo-no-slip" boundary condition because it allows the solid and fluid phases to have different velocities at the interface. Combining Eqns. (21),(23), a single vector equation for the volume-weighted mixture velocity is obtained:

$$\left[\left[\phi^s \mathbf{v}^s + \phi^f \mathbf{v}^f\right]\right] = \mathbf{0}. \tag{24}$$

We now consider the choice of element degrees of freedom that will facilitate the enforcement of the interface continuity conditions. The nodal degrees of freedom for an element have been defined as containing first the solid degrees of freedom for all nodes in the element, then the fluid degrees of freedom. The traditional assembly of these degrees of freedom along an interface would satisfy continuity of velocity, but would not satisfy the interface conditions between two dissimilar materials. By equating the solid nodal velocities at the interface, the condition that the solid velocities be continuous across a discontinuity (interface) is properly satisfied. However, the fluid velocity across a discontinuity need not be continuous, and thus the standard biphasic finite element analysis will impose an incorrect interface condition. In what follows, we define a method for properly accounting for the interface boundary conditions.

Figure 1 shows a section of a finite element mesh at an interface. Some of the terms that will be used in the discussion that follows are defined in this figure. Note that an interface element is an element that contains at least one interface node.

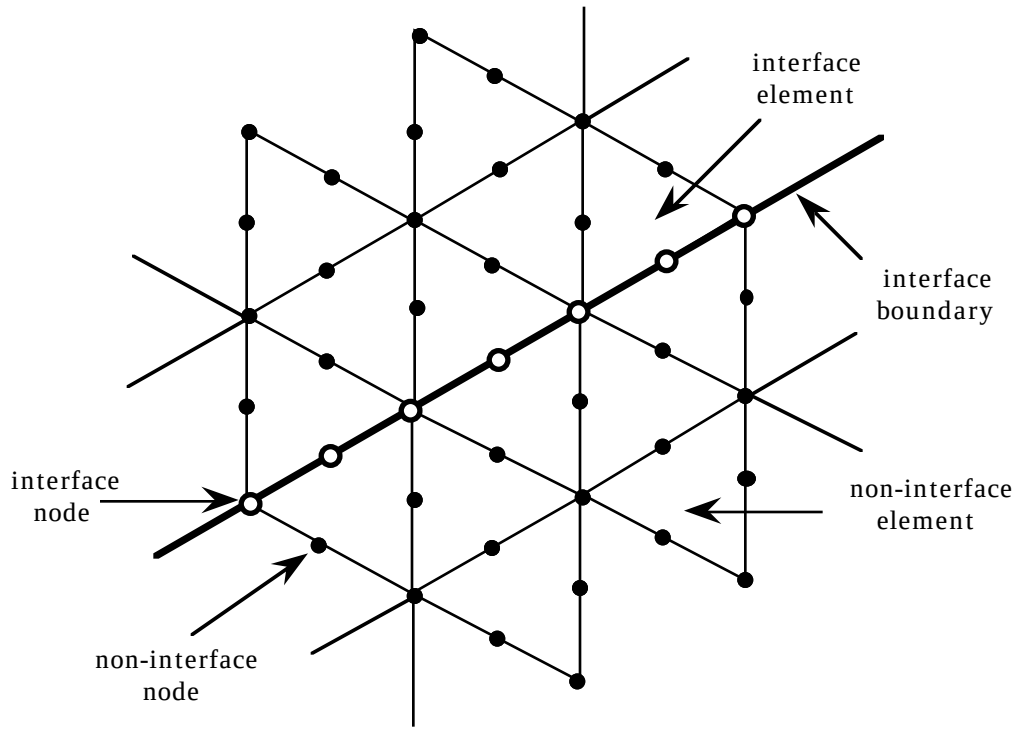

Figure 1. Portion of an arbitrary finite element mesh at an interface between two dissimilar materials. An interface element is one which contains at least one interface node. For the present formulation, we imploy a quadratic, six-node velocity interpolation and a linear, discontinuous pressure interpolation. The pressure interpolation is not directly affected by our interface methodology, and is not shown.

For purposes of this discussion the degrees of freedom for an element are reordered to contain the solid degrees of freedom for the first node, then the second, and so on, followed by a similar collection of the fluid degrees of freedom. At a node, the solid or fluid degrees of freedom are ordered according to the basis vectors, either {x,y} or {r,z} for the two dimensional case. Thus, the degrees of freedom for node 'a' of element 'e' are

$$\mathbf{v}_a^e = \begin{Bmatrix} \mathbf{v}_a^{es} \\ \mathbf{v}_a^{ef} \end{Bmatrix}. \tag{25}$$

In order to satisfy the kinematic jump condition in Eq. (24), the volume-weighted mixture velocity is introduced as a degree of freedom for nodes at an interface, replacing the fluid phase velocity degree of freedom. The degrees of freedom at the interface nodes become

$$\mathbf{v}_a^{e*} = \begin{Bmatrix} \mathbf{v}_a^{es} \\ \phi^s \mathbf{v}_a^{es} + \phi^f \mathbf{v}_a^{ef} \end{Bmatrix}. \tag{26}$$

The two sets of element degrees of freedom are related by the following transformation

$$\mathbf{v}^{e} = \mathbf{T}\mathbf{v}^{e*}, \tag{27}$$

where the transformation, **T**, is defined below. The element matrices are constructed based on the original set of degrees of freedom, then transformed according to the above relation, yielding

$$\mathbf{Q}^{*} = \mathbf{T}^{\mathbf{T}}\mathbf{Q}\mathbf{T}. \tag{28}$$

Here **Q** is an arbitrary original interface element matrix, **Q*** is the transformed element matrix expressed in terms of the new set of degrees of freedom, and **T** is defined as

$$\mathbf{T} = \left[\begin{array}{ccc|ccc} \mathbf{T}_1^{ss} & & & \mathbf{T}_1^{sf} & & \\ & \mathbf{T}_2^{ss} & & & \mathbf{T}_2^{sf} & \\ & & \ddots & & & \ddots \\ & & \mathbf{T}_{n_{en}}^{ss} & & & \mathbf{T}_{n_{en}}^{sf} \\ \hline \mathbf{T}_1^{fs} & & & \mathbf{T}_1^{ff} & & \\ & \mathbf{T}_2^{fs} & & & \mathbf{T}_2^{ff} & \\ & & \ddots & & & \ddots \\ & & \mathbf{T}_{n_{en}}^{fs} & & & \mathbf{T}_{n_{en}}^{ff} \end{array}\right], \tag{29}$$

where the submatrices are the transformations for node ‘a’ ranging from 1 to the number of element nodes, n_{en}, and the terms not listed are null. For nodes that are not on the interface, the nodal degrees of freedom do not have to be transformed. Thus, the submatrices on the main diagonal of **T** are identities and those off the diagonal are null for these nodes. For interface nodes, the submatrices are obtained from the transformation of the old nodal degrees of freedom to the new. The submatrices in the transformation will differ depending on whether the element having a node on the interface is in a region of the continuum corresponding to a biphasic material, a single phase viscous fluid, or a single phase elastic solid. If the element is in a biphasic region the matrices are

$$\mathbf{T}_a^{ss} = \mathbf{I}, \quad \mathbf{T}_a^{sf} = \mathbf{0}$$

$$\mathbf{T}_a^{fs} = \begin{bmatrix} -\dfrac{\phi^s}{\phi^f} & 0 \\ 0 & -\dfrac{\phi^s}{\phi^f} \end{bmatrix}, \quad \mathbf{T}_a^{ff} = \begin{bmatrix} \dfrac{1}{\phi^f} & 0 \\ 0 & \dfrac{1}{\phi^f} \end{bmatrix}. \tag{30}$$

For a single phase fluid region no contribution comes from the solid terms. In order to eliminate the solid contributions at fluid interface nodes all of the submatrices will be null except for the $\mathbf{T}^{ff}$ terms, which will be identities. For a single phase solid region the $\mathbf{T}^{ss}$ terms will be identities, and the remainder will be null. With the interface element matrices transformed, their contributions must be properly assembled into the global matrices and the appropriate constraints applied in order to satisfy the interface boundary conditions in Eq. (24), which includes the "pseudo-no-slip" condition.

When transformed interface degrees of freedom from the local nodes of two different elements are assembled into their global positions, special considerations are still necessary. For non-interface nodes on an interface element, or interface nodes between two dissimilar biphasic materials, the local contributions simply go to the corresponding global locations. For an interface between a biphasic material and a single-phase fluid, the contributions still go to the standard locations, but note that the fluid node is contributing zeros to the global solid degree of freedom and the biphasic node contributes the weighted mixture velocity to the global fluid degree of freedom. For an interface between solid and biphasic regions the solid velocities from each region are assembled together into the global interface degree of freedom to satisfy the continuity of the solid across the interface. The volume-weighted mixture velocity degree of freedom from the biphasic element is also assembled into the global solid degree of freedom to satisfy the jump condition of Eq. (24). This leaves the global fluid interface degree of freedom of this node with no contributions, thus this degree of freedom must be constrained to zero at the global level. For a single-phase fluid and single-phase solid interface, the no-slip condition must be also be satisfied. The fluid degree of freedom from the fluid element and the solid degree of freedom from the solid

element are each assembled into the global solid degree of freedom. By doing this, we equate the fluid velocity of the node on the fluid side of the interface with the solid velocity of the node on the solid side, thus satisfying the no-slip condition and the jump condition. The global fluid interface degree of freedom of this node is left with no contributions from the element level and therefore must be constrained to zero.

For single-phase fluid and single-phase solid, additional constraints are applied. For a single-phase fluid, all solid degrees of freedom that are not prescribed or on an interface are constrained to zero (since there is no contribution of the solid phase in this region). Similarly, the fluid degrees of freedom of a single-phase solid are also constrained to zero (since there is no contribution of the fluid phase in this region) with the exception of those that are prescribed or on an interface. With the interface boundary conditions implemented in this fashion, the finite element code can model any combination of solid/fluid/biphasic material interfaces.

EXAMPLE PROBLEMS AND RESULTS

Couette Flow over a Rigid Porous-Permeable Layer

The boundary conditions derived by Hou [10] have been used in that study to formulate the classical Taylor problem of Couette flow over a rigid, porous-permeable medium. In this problem, the steady viscous fluid flowing over and inside the porous-permeable layer is sustained by the movement of the upper boundary of the fluid layer with a speed V_0, as illustrated in Figure 2. The porous layer is assumed to be rigid and stationary and thus the velocity and displacement of the porous layer are zero. The moving plate and the porous layer are also assumed to be infinite in the x direction.

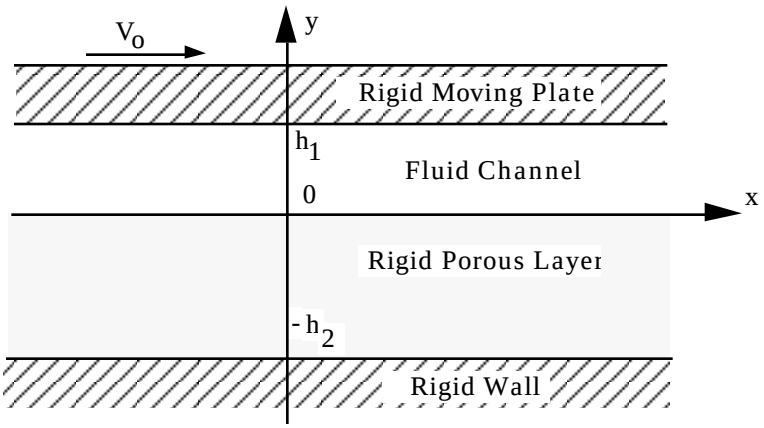

Figure 2. Schematic diagram of Couette flow over a porous-permeable layer. The finite element model will contain only the fluid channel and rigid, porous layer. Solid velocity is set to zero at all boundaries of the model. Fluid velocity is prescribed to V_0 at the top of the fluid channel, and zero at the bottom of the porous layer, and is unknown at the left and right edges.

The analytical solution of this problem states that the fluid flux depends on the following flow and geometric parameters [10]:

$$\delta = \sqrt{\frac{\mu^a}{h_2^2 K}}, \quad \xi = \frac{h_1}{h_2}, \quad \eta = \frac{\phi^{f^2}\mu^f}{\mu^a}, \tag{31}$$

where δ is a measure of viscous drag of the outside fluid relative to the diffusive drag of interstitial fluid within the porous medium, ξ is the thickness ratio of the viscous fluid layer relative to the porous solid layer and η defines a weighted viscosity ratio; μ^f is the viscosity of the fluid in the channel and μ^a is the apparent viscosity of the fluid in the porous layer. For low values of δ (< 0.02), the amount of fluid flow in the porous layer is small and the boundary layer that develops at the surface will be narrow. Therefore, for purposes of validating the finite element code the combination of material parameters chosen will be such that δ is not too small.

In this problem, the rigid porous layer will be modeled as a biphasic material (ϕ^f=0.8) with high elastic modulus (λ^s=1.0 GPa, μ^s=3.0 GPa); the viscous fluid will be modeled as a limiting case of a biphasic material (ϕ^f=1) with a viscosity of 1.0Ns/m^2 (the viscosity of synovial fluid [13]). The solid velocity is set to zero on the entire boundary, while fluid velocity is prescribed across the top of the fluid layer ($y=h_1$), unknown at the left and right edges and zero at the bottom ($y=-h_2$).

This problem assumes the porous layer and the plate to be infinite in length, therefore the length to height ratio for the geometry must be large. Since the solid displacement is assumed to be zero, only the height of the single-phase fluid layer is taken into account in determining the length-to-height ratio. The results show that this is a valid representation of the infinite length.

A finite element mesh manually graded toward the interface is used, and has been determined to produce converged solutions. Figure 3 shows the normalized fluid flux, $\phi^f v_x^f / V_0$, *versus* the dimensionless coordinate y/h_2 for the problem of Couette flow over a rigid porous solid with $\xi = 0.25$, $\eta = 1.0$ and $\delta = 0.1$. The results obtained for this problem show excellent agreement with the analytical solutions. Since the solid degrees of freedom in the porous solid are not constrained to zero in the finite element mesh, solid displacements are also obtained from the analysis. These calculated displacements, however, are negligible, in agreement with the assumption that the material is rigid.

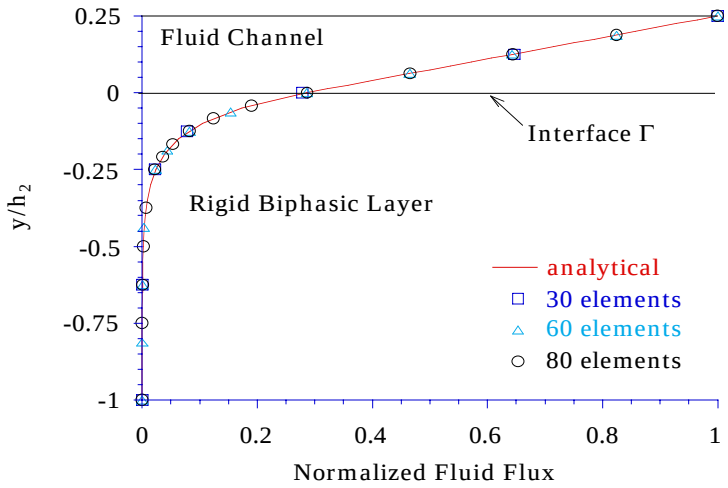

Figure 3. Finite element solution (symbols) and analytical solution (line) of the fluid flux profile in the fluid channel and in the rigid porous layer (corresponding to Figure 2) with $\eta = 1$ and $\delta = 0.1$.

Figure 4 show the finite element analyses and analytical solutions for two other cases. Though the material parameters for these cases may not model any particular material, the results

are instructive in that they show that the finite element code is valid for a wide range of parameters. The first case, with η=16.0 and δ=0.1, shows that even for a large continuous change in fluid flux across the interface, the finite element code will give excellent results. The case with η=1.0 and δ=0.02 is more representative of what will be encountered with flow over and within a biphasic soft tissue.

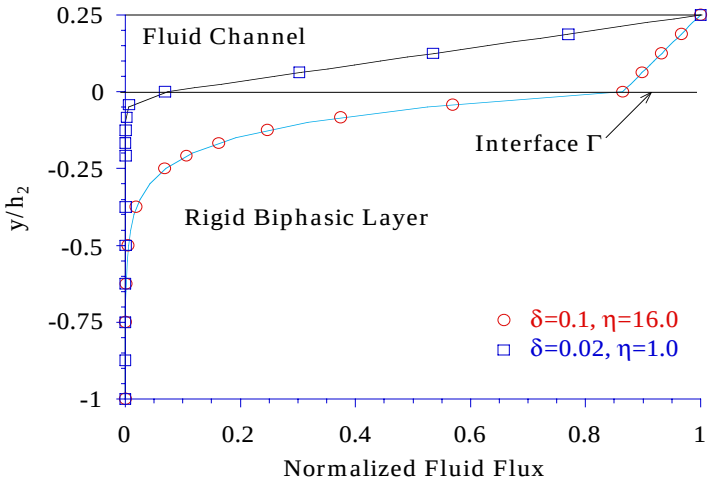

Figure 4. Finite element solution (symbols) and analytical solution (line) of the fluid flux profile in the fluid channel and in the rigid porous layer (corresponding to Figure 2) for two different cases.

Couette Flow over a Deformable Biphasic Layer

Using the boundary conditions derived by Hou, the analytical solution for Couette flow over a deformable biphasic medium has been developed [2]. An example of a deformable biphasic medium would be articular cartilage. Typical values of δ for normal cartilage are in the range of $10^{-4} - 10^{-5}$. For such small values of δ, the amount of fluid flow in the biphasic layer will be very small for the present problem and the boundary layer that develops near the surface will be extremely narrow. As a result, the code will be validated using a value of δ=0.1.

The schematic diagram of this flow problem is the same as that in Figure 2 with the rigid porous layer now being deformable. The single-phase viscous fluid is given a viscosity of

1.0Ns/m^2 (the viscosity of synovial fluid [13]). For this example the porous deformable media will be modeled as a biphasic material with the material parameters chosen so that δ=0.1 and η=1.0 (λ^s=0.1 MPa, μ^s=0.3 MPa, ϕ^s=0.2, κ=1.6×10^{-9}m^4/Ns and μ^a=0.64Ns/m^2). These properties are well within the norms for human articular cartilage [17]. The boundary conditions for this problem are as before, with the exception that the solid velocities are now unknown on the left and right edges of the model. Since the solid is deformable and the height of the biphasic layer is greater than the fluid layer, the height of the biphasic layer is used in determining the length-to-height ratio of the geometry. A ratio of 10:1 for the length to biphasic layer height is used to produce the results in Figures 5 and 6.

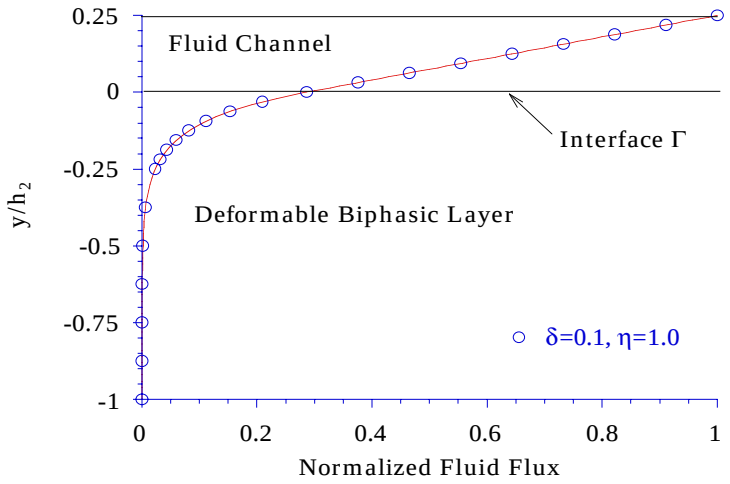

Figure 5. Finite element solution (symbols) and analytical solution (line) of the normalized fluid flux profile in the fluid channel and in porous medium with η = 1.0 and δ = 0.1.

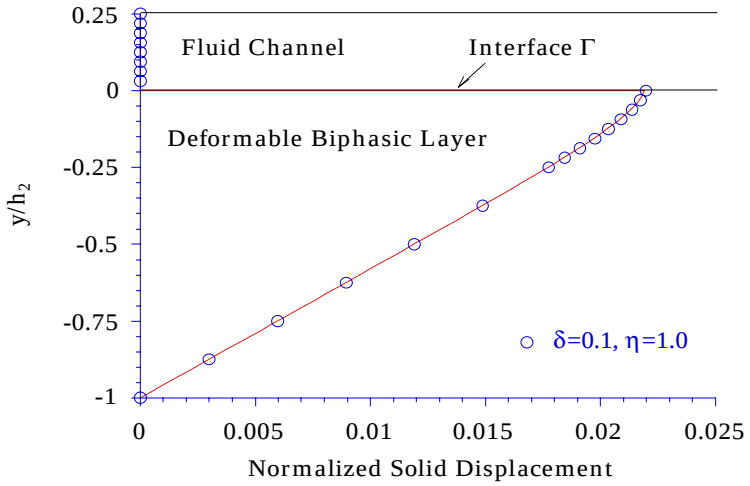

Figure 6. Finite element solution (symbols) and analytical solution (line) of the normalized solid displacement profile of the deformable porous medium with $\eta = 1.0$ and $\delta = 0.1$.

Figures 5 and 6 show the results obtain for the Couette flow over a deformable porous-permeable medium. Figure 5 shows the normalized velocity profile, $\phi^f v_x^f / V_0$, in the fluid channel and in the biphasic medium. Figure 6 show the normalized solid displacement profile of the deformable biphasic material. In this problem the normalized displacement is defined as the solid displacement divided by the height of the deformable biphasic material, d^s/h_2. The finite element solution for both the fluid flux and solid displacement agree with the analytical solution.

SUMMARY AND CONCLUDING REMARKS

The fluid viscosity and interface boundary conditions have been added to the mixed-penalty formulation of the linear biphasic theory for soft tissue and have been implemented into the finite element code. The purpose of adding viscosity to the fluid phase is to be able to model a single-phase, viscous fluid as a limiting case of a biphasic material. The addition of interface boundary conditions allows us to model problems involving viscous incompressible fluids, elastic solids, and biphasic materials, such as would be found in a diarthrodial joint. A volume-weighted mixture velocity is introduced as a nodal degree of freedom for nodes on an interface. With proper

treatment of the constraints and of the assembly of elements, the interface kinematic boundary conditions set forth by Hou [10] are satisfied exactly in the finite element solution.

The finite element code can now model any combination of interfaces involving viscous incompressible fluids, elastic solids and biphasic materials. This is a first step toward modeling the mechanisms of joint lubrication. Consideration should now be given to solving problems representative of diarthrodial joint lubrication. For example, the problem of two opposing biphasic layers separated by a layer of viscous incompressible synovial fluid is fundamental to the understanding of joint lubrication. The finite element program can be used to solve this problem up to the time when the biphasic tissues come into contact, after which the biphasic contact approach of Donzelli and Spilker [6] must be used.

ACKNOWLEDGEMENTS

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FIGURE LEGENDS

Figure 1. Portion of a finite element mesh at an interface between two dissimilar materials. An interface element is one which contains at least one interface node.

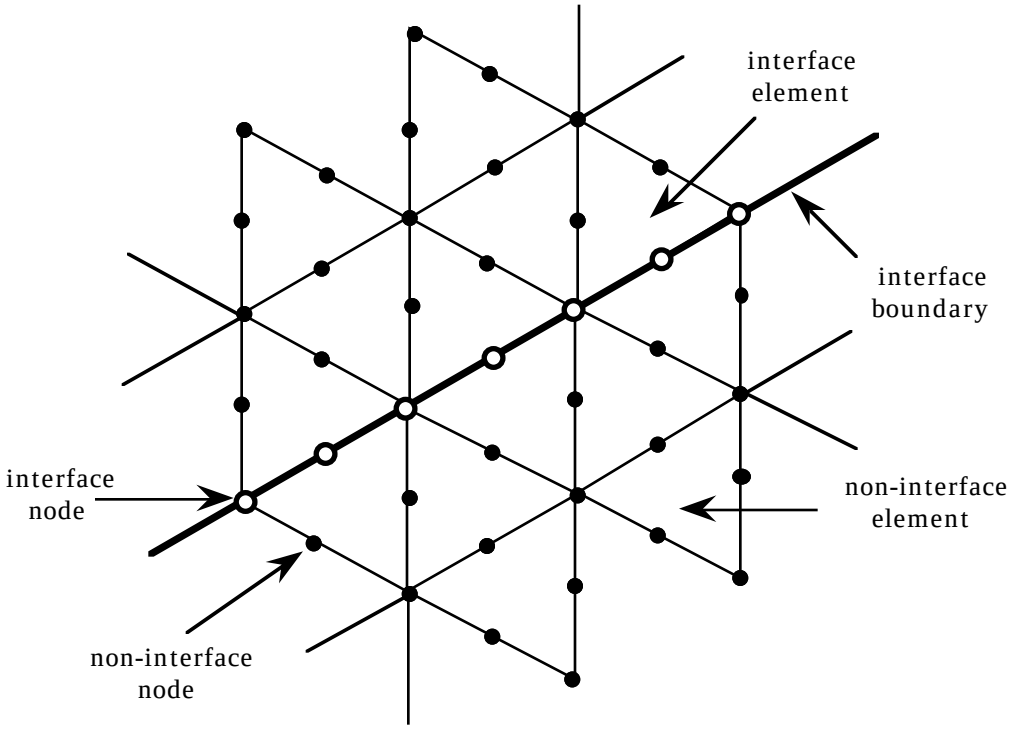
Figure 2. Schematic diagram of Couette flow over a porous-permeable layer. The finite element model will contain only the fluid channel and rigid, porous layer. Solid velocity is set to zero at all boundaries of the model. Fluid velocity is prescribed to V_0 at the top of the fluid channel, and zero at the bottom of the porous layer, and is unknown at the left and right edges.

Figure 3. Finite element solution (symbols) and analytical solution (line) of the fluid flux profile in the fluid channel and in the rigid porous layer (corresponding to Figure 2) with $\eta = 1$ and $\delta = 0.1$.

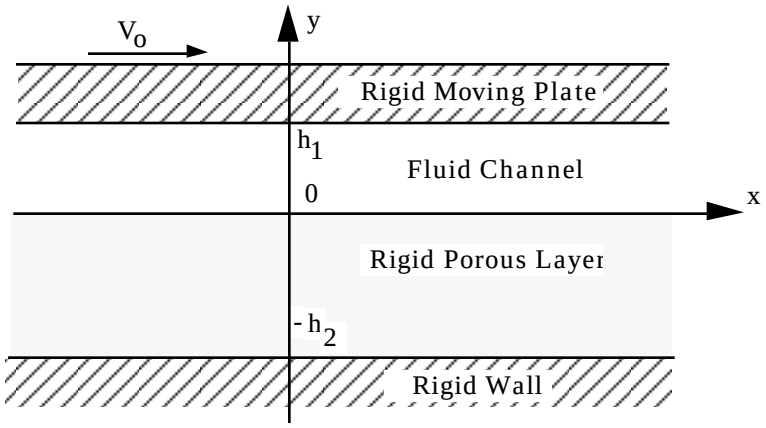
Figure 4. Finite element solution (symbols) and analytical solution (line) of the fluid flux profile in the fluid channel and in the rigid porous layer (corresponding to Figure 2) for two different cases.

Figure 5. Finite element solution (symbols) and analytical solution (line) of the normalized fluid flux profile in the fluid channel and in porous medium with $\eta = 1.0$ and $\delta = 0.1$.

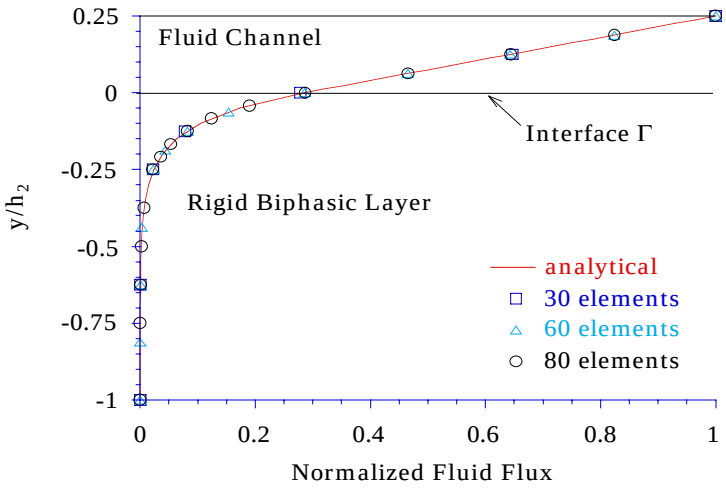
Figure 6. Finite element solution (symbols) and analytical solution (line) of the normalized solid displacement profile of the deformable porous medium with $\eta = 1.0$ and $\delta = 0.1$.


interface
element
interface
boundary
interface
node
non-interface
element
non-interface
node

1


V_o
y
Rigid Moving Plate
h_1
Fluid Channel
0
x
Rigid Porous Layer
$-h_2$
Rigid Wall

1


0.25
0
-0.25
-0.5
-0.75
-1
y/h2
Fluid Channel
Interface Γ
Rigid Biphasic Layer
analytical
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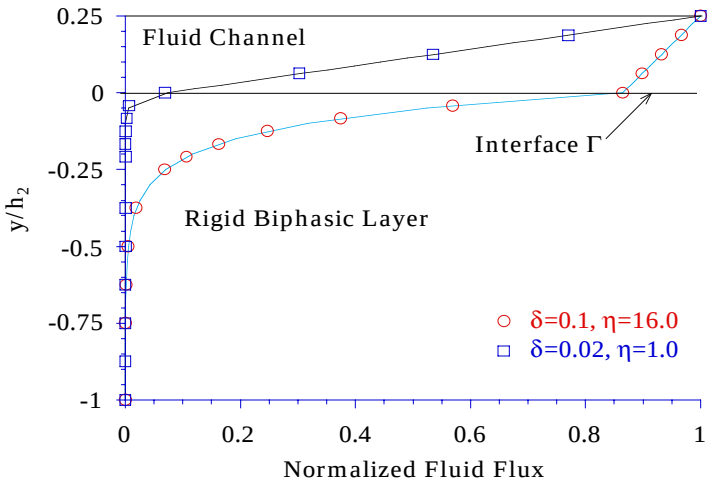
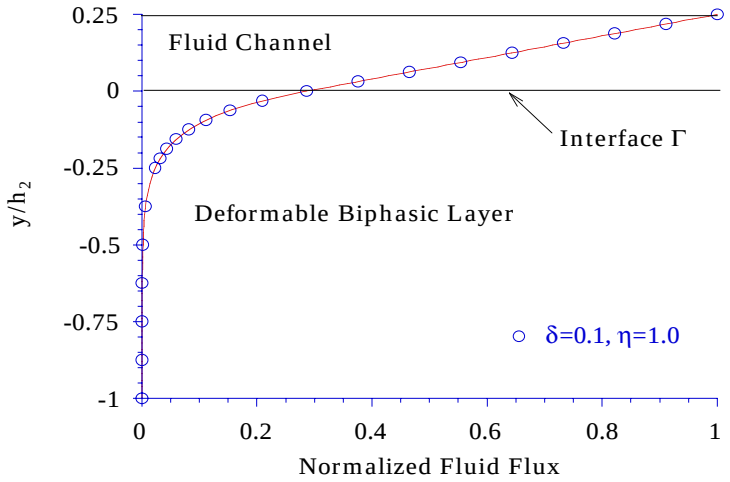
60 elements
80 elements
0
0.2
0.4
0.6
0.8
1
Normalized Fluid Flux

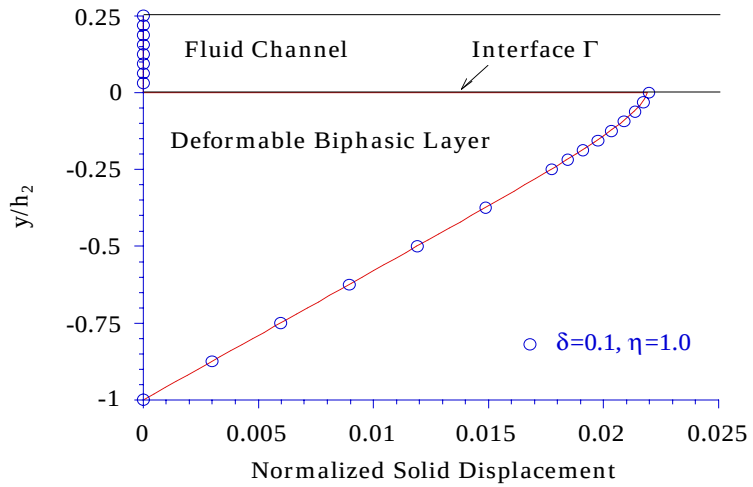
1


0.25
0
-0.25
-0.5
-0.75
-1
y/h2
Fluid Channel
Interface Γ
Rigid Biphasic Layer
δ=0.1, η=16.0
δ=0.02, η=1.0
0
0.2
0.4
0.6
0.8
1
Normalized Fluid Flux

1


0.25
0
-0.25
-0.5
-0.75
-1
y/h2
Fluid Channel
Interface Γ
Deformable Biphasic Layer
δ=0.1, η=1.0
0
0.2
0.4
0.6
0.8
1
Normalized Fluid Flux

1


0.25
0
-0.25
-0.5
-0.75
-1
y/h2
Fluid Channel
Interface Γ
Deformable Biphasic Layer
δ=0.1, η=1.0
0
0.005
0.01
0.015
0.02
0.025
Normalized Solid Displacement

1

2