A POSTERIORI ERROR ESTIMATION FOR DISCONTINUOUS GALERKIN SOLUTIONS OF HYPERBOLIC PROBLEMS

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Abstract

We analyze the spatial discretization errors associated with solutions of one-dimensional hyperbolic conservation laws by discontinuous Galerkin methods in space. We show that the leading term of the spatial discretization error with piecewise polynomial approximations of degree p is proportional to a Radau polynomial of degree $p+1$ on each element. We also prove that the local and global discretization errors are $O(\Delta x^{2(p+1)})$ and $O(\Delta x^{2p+1})$ at the downwind point of each element. This strong superconvergence enables us to show that local and global discretization errors converge as $O(\Delta x^{p+2})$ at the remaining roots of Radau polynomial of degree $p+1$ on each element. Convergence of local and global discretization errors to the Radau polynomial of degree $p+1$ also holds for smooth solutions as $p \to \infty$. These results are used to construct asymptotically correct *a posteriori* estimates of spatial discretization errors that are effective for linear and nonlinear conservation laws in regions where solutions are smooth.

1 Introduction

The discontinuous Galerkin method (DGM) provides an appealing approach to address problems having discontinuities, such as those that arise in hyperbolic conservation laws. Originally developed for neutron transport problems [20], the DGM has been used to solve ordinary differential equations [21, 18] and hyperbolic [6, 7, 8, 9, 13, 11, 14, 17],

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parabolic [15, 16], and elliptic [5, 4, 24] partial differential equations. These citations only represent a small sampling of the work that has been done on DGMs. The proceedings of Cockburn *et al.* [10] contains a more complete and current survey of the method and its applications.

The DGM may be regarded as a way of extending finite volume methods to arbitrarily high orders of accuracy. The solution space is a piecewise continuous (polynomial) function relative to a structured or unstructured mesh. As such, it can sharply capture solution discontinuities relative to the computational mesh. It maintains local conservation on an elemental basis. Regardless of order, the DGM has a simple communication pattern to elements with a common face that makes it useful for parallel computation. It can handle problems in complex geometries to high order. And, it is useful with adaptivity since interelement continuity is neither required for h-refinement (mesh refinement and coarsening) nor p-refinement (method order variation).

In order for the DGM to be useful in an adaptive setting, techniques for estimating the discretization errors should be available both to guide adaptive enrichment and to provide a stopping criteria for the solution process. Focusing on one-dimensional hyperbolic conservation laws

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \mathbf{0}, \tag{1}$$

where $\mathbf{u}(x,t)$ and $\mathbf{f}(\mathbf{u})$ are m-vectors, t is time, and x is a spatial coordinate, we note that Bey and Oden [6] and Süli *et al.* [22, 23] studied *a posteriori* error estimates for linear problems. Using results of Adjerid *et al.* [2], Biswas *et al.* [9] hypothesized that spatial DGM solutions of (1) using piecewise-continuous polynomials of degree p exhibited superconvergence at Radau points of degree $p+1$ on each element. The Radau points are the roots of the sum or difference of successive Legendre polynomials of degrees p and $p+1$ (§2). The choice of sign fixes one root at the left or right end of each element. The appropriate selection for (1) is the downwind end of the element relative to component velocities determined from a diagonalization of the Jacobian $\mathbf{f}_{\mathbf{u}}(\mathbf{u})$.

Biswas *et al.* [9] used this superconvergence to construct *a posteriori* estimates of spatial discretization errors of (1) by assuming that the error on each element behaved like a Radau polynomial of degree $p+1$. Cockburn *et al.* [12] also used the superconvergence to extract higher-order solutions to (1) through postprocessing. Computational results in both cases were excellent, but there was no formal analysis to verify that superconvergence does actually occur at the Radau points. Karakashian and Makridakis [19] overcame this by analyzing a space-time DGM for the nonlinear Schrodinger equation and showing that the solution is locally superconvergent in time at Radau points.

By analyzing one-dimensional problems, we verify that the Radau points are indeed superconvergence points of the DGM. If Δx is characteristic of the mesh spacing, the local discretization errors of piecewise-polynomial-degree-p DGM solutions converge at an $O(\Delta x^{p+1})$ rate. We show (§2) that local errors at Radau points are $O(\Delta x^{p+2})$ with the exception of the Radau point at the downwind end of the element, which converges at an $O(\Delta x^{2(p+1)})$ rate. We obtain an explict series representation for the local discretization error with the Radau polynomial of degree $p+1$ as its leading term. This representation enables us to establish a form of superconvergence in p at the downwind point in addition to the superconvergence in Δx. In particular, when solutions are smooth, we show that

the local error at the downwind point of each element converges at exponential rate in p relative to the L^2 error on the element.

Superconvergence of local errors is not sufficient to explain the success of the global *a posteriori* error estimates of Biswas *et al.* [9]. However, the high convergence rate at the downwind end of each element enables us to show that global discretization errors converge as $O(\Delta x^{2p+1})$ at the downwind point while maintaining an $O(\Delta x^{p+2})$ rate at the remainder of the Radau points (§2). The global error behavior at the downwind point had been shown correct by Le Saint and Raviart [21] who utilized an equivalence between the DGM an implicit Runge-Kutta method with collocation at the Radau points. Our analysis is applicable to both linear (§2.1) and nonlinear (§2.2) problems with smooth solutions. As with the local error estimates, we obtain an explicit series representation for the global discretization error with the leading term proportional to the Radau polynomial of degree $p+1$. This behavior enables us to develop *a posteriori* error estimates which are applied to several examples (§3).

2 Error Analyses

Focusing on the spatial discretization error of the DGM, we assume that time integration is exact and analyze several ordinary differential systems that are related to (1). To begin, we consider the linear, scalar, initial value problem

$$u' - au = 0, \qquad x > 0, \qquad u(0) = u_0, \tag{2}$$

where $(\ )' = d(\ )/dx$. In order to construct a DGM solution of (2), (i) introduce a subdivision of x with $x_n = n\Delta x$, $n = 0, 1, \ldots$; (ii) select an approximation $U(x)$ of $u(x)$ such that the restriction $U_n(x)$ of $U(x)$ to $(x_{n-1}, x_n]$ is an element of $\mathcal{P}_p$, which consists of polynomials of degree p in $x \in (x_{n-1}, x_n]$; and (iii) satisfy

$$[U_n(x_{n-1}^+) - U_{n-1}(x_{n-1}^-)]V(x_{n-1}) + \int_{x_{n-1}}^{x_n} V(x)[U_n'(x) - aU_n(x)]dx = 0, \qquad \forall\, V \in \mathcal{P}_p, \tag{3}$$

where $U(x^\pm) = \lim_{\delta\to 0} U(x \pm \delta)$ and $U_0(0^-) = u_0$. This formulation may be obtained by multiplying (2) by a test function V, integrating on (x_{n-1}^-, x_n^-), and accounting for the discontinuous trial space. The exact solution of (2) on (x_{n-1}, x_n) has no jumps at x_{n-1} and satisfies the standard Galerkin form

$$\int_{x_{n-1}}^{x_n} V(x)[u'(x) - au(x)]dx = 0, \qquad \forall\, V \in \mathcal{P}_p. \tag{4}$$

Let us examine the local discretization error

$$\epsilon(x) = u(x) - U_n(x) \tag{5}$$

on (x_{n-1}, x_n). In this case, $U_{n-1}(x_{n-1}^-) = u(x_{n-1})$ and a subtraction of (3) from (4) reveals the "Galerkin orthogonality" relation

$$\epsilon(x_{n-1})V(x_{n-1}) + \int_{x_{n-1}}^{x_n} V(\epsilon' - a\epsilon)dx = 0, \qquad \forall\, V \in \mathcal{P}_p. \tag{6}$$

Transforming (x_{n-1}, x_n) to the canonical element $(-1,1)$ by the linear mapping

$$x(\xi) = \frac{x_n + x_{n-1}}{2} + \xi \frac{\Delta x}{2} \tag{7a}$$

and integrating (6) by parts yields

$$\hat{\epsilon}(1)\hat{V}(1) - \int_{-1}^{1} \hat{\epsilon}(\hat{V}' + \frac{a\Delta x}{2}\hat{V})d\xi = 0, \qquad \forall\ \hat{V} \in \mathcal{P}_p, \tag{7b}$$

where $\hat{\epsilon}(\xi) = \epsilon(x(\xi))$, *etc.*, and the domain of $\mathcal{P}_p$ is now understood as being $(-1,1)$. (When there is no confusion, we will omit the $\hat{}$ on ϵ and V.)

Recall [1], that the Legendre polynomials $P_k(\xi)$, $k = 0, 1, \ldots$, satisfy the Rodrigues formula

$$P_k(\xi) = \frac{1}{(-1)^k 2^k k!} \frac{d^k}{d\xi^k}(1-\xi^2)^k = \frac{(2k)!}{2^k (k!)^2}\xi^k + \sum_{j=0}^{k-1} a_j \xi^j, \qquad k \geq 0, \tag{8a}$$

and the orthogonality relation

$$\int_{-1}^{1} P_k P_j(\xi) d\xi = \frac{2\delta_{kj}}{2k+1}, \qquad j,k \geq 0, \tag{8b}$$

where δ_{kj} is the Kronecker delta. With these definitions, the right Radau polynomial $R_k(\xi)$ of degree k satisfies

$$R_k(\xi) = \begin{cases} P_k(\xi) - P_{k-1}(\xi), & \text{if } k > 0 \\ 0, & \text{if } k = 0 \end{cases}. \tag{8c}$$

Since $P_j(1) = 1$, $j = 0, 1, \ldots$ [1], we see that $R_k(1) = 0$, $k = 0, 1, \ldots$. A left Radau polynomial would be the sum of two consecutive Legendre polynomials and it would always have a root at $\xi = -1$. Both left and right Radau polynomials will be needed to construct error estimates for hyperbolic systems; however, we focus on right Radau polynomials for the simplified problem (2).

With these preliminary results, we show that the local discretization error is asymptotically proportional to $R_{p+1}(\xi)$.

Theorem 1. *Let $U_n(x) \in \mathcal{P}_p$, $p \geq 0$, satisfy (3) with $U(x_{n-1}^-) = u(x_{n-1})$, then*

$$\epsilon(\xi) = \sum_{k=p+1}^{\infty} Q_k(\xi)\Delta x^k, \tag{9a}$$

$$\epsilon(1) \;=\; Q_{2p+2}(1)\Delta x^{2p+2} + O(\Delta x^{2p+3}) = \alpha_{p+1}\frac{(-a)^{p+1}p!}{(2p+1)!}\Delta x^{2p+2} + O(\Delta x^{2p+3}), \tag{9b}$$

with $Q_k(\xi) \in \mathcal{P}_k$, $k \geq p+1$, satisfying

$$Q_{p+1}(\xi) = \alpha_{p+1} R_{p+1}(\xi), \tag{9c}$$

$$Q_k(1) = 0, \qquad k = p+1, p+2, \ldots, 2p+1, \tag{9d}$$

and $\alpha_{p+1} \in \Re$.

Proof. Since $u, U_n \in C^\infty(x_{n-1}, x_n)$, ϵ has a series expansion in powers of Δx. Terms of order Δx^k, $k = 0, 1, \ldots, p$, of this series vanish with a basis of polynomials of degree p. This may be shown by using the orthogonality properties (8b) of the Legendre polynomials; however, we omit this construction because the arguments essentially duplicate those to follow. Thus, (9a) exists and its substitution into (7b) and a subsequent grouping of terms having common powers of Δx yields

$$\left[Q_{p+1}(1)V(1) - \int_{-1}^{1} Q_{p+1}V' d\xi\right] \Delta x^{p+1} +$$

$$\sum_{k=p+2}^{\infty} \left[Q_k(1)V(1) - \int_{-1}^{1} (Q_k V' + \frac{aQ_{k-1}V}{2}) d\xi\right] \Delta x^k = 0, \qquad \forall\ V \in \mathcal{P}_p. \tag{10}$$

The leading ($O(\Delta x^{p+1})$) term of (10) satisfies

$$Q_{p+1}(1)V(1) - \int_{-1}^{1} Q_{p+1}V' d\xi = 0, \qquad \forall\ V \in \mathcal{P}_p. \tag{11}$$

This with $V(\xi) = \xi^j$, $j = 0, 1, \ldots, p$, yields

$$Q_{p+1}(1) = 0 \tag{12a}$$

and

$$\int_{-1}^{1} Q_{p+1}(\xi)\xi^j d\xi = 0, \qquad j = 1, 2, \ldots, p-1. \tag{12b}$$

Expanding both $Q_{p+1}(\xi)$ and ξ^j in series of Legendre polynomials and using the orthogonality relation (8b) with (12a) yields (9c).

We use an induction argument on k in (10) to show that

$$Q_k(\xi) = \sum_{j=2(p+1)-(k+1)}^{k} c_{kj} P_j(\xi), \qquad k \geq p+1. \tag{13}$$

Having established (9c), we see that (13) is satisfied with $k = p+1$.

The $O(\Delta x^k)$ term in (10) satisfies

$$Q_k(1)V(1) - \int_{-1}^{1} (Q_k V' + \frac{aQ_{k-1}V}{2}) d\xi = 0, \qquad \forall\ V \in \mathcal{P}_p. \tag{14}$$

Choosing $V(\xi) = 1$ yields

$$Q_k(1) = \frac{a}{2} \int_{-1}^{1} Q_{k-1}(\xi) d\xi. \tag{15}$$

Using the induction hypothesis (13) with k replaced by $k-1$ and (8b) leads to the superconvergence result

$$Q_k(1) = 0, \qquad k < 2(p+1). \tag{16}$$

Next, select $V(\xi) = \xi^j$ in (14) and use (16) to obtain

$$j \int_{-1}^{1} Q_k \xi^{j-1} d\xi + \frac{a}{2} \int_{-1}^{1} Q_{k-1} \xi^j d\xi = 0, \qquad \begin{aligned} j &= 0, 1, \ldots, 2(p+1) - (k+1), \\ k &= p+2, \ldots, 2p+1. \end{aligned} \tag{17}$$

The second integral vanishes with the induction hypothesis (13) and the orthogonality condition (8b); thus, $Q_k(\xi)$ is orthogonal to all polynomials of degree $2(p-1) - (k+2)$ or less. This establishes (13).

Finally, to prove (9b), we use (15) to conclude that $Q_{2(p+1)}(1)$ is the first nonvanishing term in the series (9a) evaluated at $\xi = 1$. Using (15) and (17) with $k = 2(p+1)$ we obtain

$$Q_{2(p+1)}(1) = \frac{a}{2} \int_{-1}^{1} Q_{2p+1} d\xi = \frac{(-1)^p a^{p+1}}{2^{p+1} p!} \int_{-1}^{1} Q_{p+1} \xi^p d\xi. \tag{18}$$

Using (8a), (8b), and (9c) yields

$$Q_{2(p+1)}(1) = \alpha_{p+1} \frac{(-a)^{p+1} p!}{(2p+1)!}. \tag{19}$$

Combining this and (9a), (9c), and (9d) establishes (9b) and completes the analysis. □

Thus, locally, the solution converges at an $O(\Delta x^{2(p+1)})$ rate at the "downwind" end ($\xi = 1$) of the subinterval and at an $O(\Delta x^{p+2})$ rate at the remainder of the Radau points. Elsewhere, the local error is $O(\Delta x^{p+1})$. We can also demonstrate superconvergence in p at the downwind end of each element.

Theorem 2. *Under the conditions of Theorem 1,*

$$\frac{|\epsilon(x_n)|}{\|\epsilon\|_{\mathcal{L}_{2,n}}} \leq C \frac{\Delta x^p}{p^p} |G(\cdot, x_n)|_{p+1,n} \qquad p \to \infty, \qquad \Delta x \to 0, \tag{20}$$

where $\|\cdot\|_{\mathcal{L}_{2,n}}$ *is the* $\mathcal{L}_2$ *norm and* $|\cdot|_{p+1,n}$ *is the Sobolev* $\mathcal{H}_{p+1}$ *seminorm on* (x_{n-1}, x_n) *.*

Proof. Consider the fundamental solution of (2)

$$G(x, \bar{x}) = e^{a(\bar{x}-x)} H(\bar{x} - x), \tag{21}$$

where $H(x)$ is the Heaviside function and $G(x, \bar{x})$ satisfies

$$\frac{\partial G(x, \bar{x})}{\partial x} + a G(x, \bar{x}) = 0, \qquad x < \bar{x}. \tag{22}$$

Multiplying (22) by $\epsilon(x)$, setting $\bar{x} = x_n$, integrating on $[x_{n-1}, x_n]$, and integrating by parts yields

$$\epsilon(x_n) = \epsilon(x_{n-1}) G(x_{n-1}, x_n) + \int_{x_{n-1}}^{x_n} (\epsilon' - a\epsilon) G(\cdot, x_n) dx. \tag{23}$$

Subtracting (6) and integrating the result by parts yields

$$\epsilon(x_n) = -\int_{x_{n-1}}^{x_n} \epsilon[(G(\cdot, x_n) - V(\cdot))' + a(G(\cdot, x_n) - V(\cdot))] dx, \qquad \forall V \in \tilde{V}, \tag{24}$$

where $\tilde{V} = \{V \in \mathcal{P}_p \mid V(x_n) = G(x_n, x_n)\}$. Using Schwarz' inequality we obtain

$$\frac{|\epsilon(x_n)|}{\|\epsilon\|_{\mathcal{L}_{2,n}}} < C\|G(\cdot, x_n) - V\|_{\mathcal{H}_{1,n}}, \qquad \forall V \in \tilde{V}. \tag{25}$$

Using standard interpolation error estimates [3] as $p \to \infty$ leads to (20). □

Thus, convergence at x_n is exponential relative to the $\mathcal{L}^2$ error on the element, which also exhibits exponential convergence in p.

The previous information about the local error with the following two lemmas describing the absolute stability and the strong superconvergence at downwind element ends can be used to estimate the global error.

Lemma 1. *Let $U_n(x) \in \mathcal{P}_p$ be a solution of (3) then*

$$U_n(x_n) = \Gamma(z) U_{n-1}(x_{n-1}) \tag{26a}$$

where

$$z = a\Delta x, \qquad \Gamma(z) = e^z + O(z^{2p+2}). \tag{26b}$$

Proof. cf. Le Saint and Raviart [21]. □

Lemma 2. *Let $U_n(x) \in \mathcal{P}_p$ be a solution of (3) then the global error*

$$e_n(x) = u(x) - U_n(x), \qquad x \in (x_{n-1}, x_n], \tag{27a}$$

satisfies

$$e_n(x_n) = O(\Delta x^{2p+1}). \tag{27b}$$

Proof. Let $\tilde{U}_n(x)$, $x \in (x_{n-1}, x_n]$, be the solution of (3) satisfying the initial condition $U(x_{n-1}^-) = u(x_{n-1})$, then using (26)

$$e_n(x_n) = u(x_n) - \tilde{U}_n(x_n) + \tilde{U}_n(x_n) - U_n(x_n) = \epsilon(x_n) + \Gamma(z) e_{n-1}(x_{n-1}). \tag{28a}$$

Iterating this recurrence and approximating the exponential in (26b) yields

$$e_n(x_n) = \sum_{i=0}^{n} \Gamma(z)^i \epsilon(x_{n-i}) = \sum_{i=0}^{n} [1 + iz + O((iz)^2)] \epsilon(x_{n-i}). \tag{28b}$$

Summing the series while using (9b) and (26b) with $n = O(1/\Delta x)$ yields the desired result (27b). □

The strong superconvergence at $x = x_n$ implied by (27b) enables us to prove that the leading term of the global discretization error is also proportional to $R_{p+1}(\xi)$, $p > 0$. Lacking this superconvergence when $p = 0$, we will not be able to estimate the global errors of piecewise-constant discontinuous Galerkin solutions.

Theorem 3. *Let $U_n(x) \in \mathcal{P}_p$, $p > 0$, be the solution of (3) then*

$$e_n(\xi) = \sum_{k=p+1}^{2p} Q_{k,n}(\xi)\Delta x^k + O(\Delta x^{2p+1}), \tag{29a}$$

with

$$Q_{p+1,n}(\xi) = \alpha_{p+1,n} R_{p+1}(\xi) \tag{29b}$$

and

$$Q_{k,n}(1) = 0, \qquad k = p+1, p+2, \ldots, 2p. \tag{29c}$$

Proof. As in Theorem 1, the smoothness of U_n for (2) implies the existence of the series (29a). Let us add and subtract $u(x_{n-1})V(x_{n-1})$ to (3) and subtract the result from (4) to obtain

$$e_n(x_{n-1}^+)V(x_{n-1}) - e_{n-1}(x_{n-1}^-)V(x_{n-1}) + \int_{x_{n-1}}^{x_n} V(e' - ae)dx \;=\; 0, \qquad \forall\, V \in \mathcal{P}_p. \tag{30}$$

Integrating by parts and using the linear transformation (7a) yields

$$e_n(1^-)V(1) = e_{n-1}(1^-)V(1) + \int_{-1}^{1} e_n(V' + \frac{a\Delta x}{2}V)d\xi, \qquad \forall\, V \in \mathcal{P}_p. \tag{31}$$

Using (29a) and (27b) in (31) while collecting terms having like powers of Δx and following the reasoning used to prove (9a) yields

$$\left(Q_{p+1,n}(1)V(1) - \int_{-1}^{1} Q_{p+1,n}V'd\xi\right)\Delta x^{p+1} +$$

$$\sum_{k=p+2}^{\infty}\left(Q_{k,n}(1)V(1) - \int_{-1}^{1}[Q_{k,n}V' + a\frac{Q_{k-1,n}V}{2}]d\xi\right)\Delta x^k \;-\; V(1)O(\Delta x^{2p+1}) = 0,$$

$$\forall\, V \in \mathcal{P}_p. \tag{32}$$

Continuing to follow the reasoning of Theorem 1 establishes (29b) and (29c). □

The analysis can be applied to linear hyperbolic equations of the form (1) with $f(u) \propto u$ by using a Fourier transform in time to dimensionally reduce the partial differential equation to one having the form (2). The results of Theorems 1 and 3 would then apply to the spatial discretization errors.

2.1 Linear systems

The error analysis readily extends to linear vector initial value problems of the form

$$\mathbf{u}' - \mathbf{A}\mathbf{u} = 0, \qquad x > 0, \qquad \mathbf{u}(0) = \mathbf{u}_0, \tag{33}$$

where $\mathbf{A}$ is an $m \times m$ real matrix and $\mathbf{u}$ is an m-vector.

Theorem 4. *Let $\mathbf{U}_n(x) \in (\mathcal{P}_p)^m$ be the solution of the discontinuous Galerkin form of (33)*

$$\mathbf{V}^T(x_{n-1})[\mathbf{U}_n(x_{n-1}^+) - \mathbf{U}_{n-1}(x_{n-1}^-)] + \int_{x_{n-1}}^{x_n} \mathbf{V}(x)^T[\mathbf{U}_n'(x) \quad - \quad \mathbf{A}\mathbf{U}_n(x)]dx = 0,$$
$$\forall\ \mathbf{V} \in (\mathcal{P}_p)^m, \qquad (34)$$

then the local and global error estimates of Theorems 1 and 3, respectively, apply with $\boldsymbol{\alpha}_{p+1}$, $\boldsymbol{\alpha}_{p+1,n}$, $\mathbf{Q}_k$, and $\mathbf{Q}_{k,n} \in \Re^m$, $k = p+1, p+2, \ldots$.

Proof. The proof follows the scalar analyses with arguments being used on a component level. □

Remark. In many cases, one might factor $\mathbf{A}$ as

$$\mathbf{SAS}^{-1} = \begin{bmatrix} \boldsymbol{\Lambda}_+ & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_- \end{bmatrix} \qquad (35)$$

where $\boldsymbol{\Lambda}_\pm$ have eigenvalues with positive and negative real parts, respectively. Using the transformation $\mathbf{w} = \mathbf{Su}$ reduces (33) to a block diagonal form that could be used for two-point boundary value problems. If the magnitudes of $\boldsymbol{\Lambda}_\pm$ are large, (33) would be stable when initial data is prescribed at the left endpoint for the $\boldsymbol{\Lambda}_+$ part of the system and terminal data is prescribed at the right endpoint for the $\boldsymbol{\Lambda}_-$ part of the system. In this case, numerical stability would be enhanced by using Radau polynomials with a root at $\xi = 1$ would be appropriate for the initial value portion of the system while those with a root at $\xi = -1$ would be appropriate for the terminal value part. The analysis of these problems, which are important for the partial differential systems (1), also follows the scalar results of Theorems 1 and 3.

2.2 Nonlinear problems

The error analysis of nonlinear scalar and vector initial value problems having smooth solutions is, likewise, similar to the previous analyses. We present results for a scalar problem of the form

$$u' - f(u) = 0, \qquad x > 0, \qquad u(0) = u_0, \qquad (36)$$

where $f(u) \in C^{2p+1}$. The DGM for this problem is

$$[U_n(x_{n-1}^+) - U_{n-1}(x_{n-1}^-)]V(x_{n-1}) + \int_{x_{n-1}}^{x_n} V(x)[U_n'(x) \quad - \quad f(U_n(x))]dx = 0,$$
$$\forall\ V \in \mathcal{P}_p. \qquad (37)$$

Theorem 5. *Let $U_n(x) \in \mathcal{P}_p$ be a solution of (37) with $U(x_{n-1}^-) = u(x_{n-1})$ and $f(u) \in C^{2p+1}$, then the local error (5) satisfies (9c,d) and*

$$\epsilon(\xi) = \sum_{k=p+1}^{2p+1} Q_k(\xi)\Delta x^k + O(\Delta x^{2p+2}), \qquad (38a)$$

with

$$\epsilon(1) = O(\Delta x^{2(p+1)}). \tag{38b}$$

Proof. As in the linear case, subtracting a continuous Galerkin problem on (x_{n-1}, x_n) from (37), setting $U_{n-1}(x_{n-1}^-) = u(x_{n-1})$ for a local error analysis, transforming the resulting approximation to $(-1, 1)$ using (7a), defining the error ϵ according to (5), and integrating by parts yields

$$\epsilon(1)V(1) - \int_{-1}^{1} \{\epsilon V' + \frac{\Delta x}{2}[f(u) - f(U_n)]V\} d\xi = 0, \qquad \forall\ V \in \mathcal{P}_p. \tag{39}$$

Using Taylor's series expansions

$$f(u) - f(U_n) = \epsilon a(u) - \frac{\epsilon^2}{2}\frac{d^2 f(\tilde{u})}{du^2}, \tag{40a}$$

with $\tilde{u} \in (u, U_n)$ and

$$a(u) = \frac{df(u(x))}{du} = \sum_{j=0}^{2p} 2a_j(\xi)\Delta x^j + O(\Delta x^{2p+1}), \qquad a_j(\xi) = \frac{(1+\xi)^j}{2^{j+1}j!}\frac{d^j a(-1)}{dx^j}, \quad \xi \in [-1, 1]. \tag{40b}$$

The transformation (7a) was used to obtain (40b). With (40a,b), (39) becomes

$$\epsilon(1)V(1) - \int_{-1}^{1} [\epsilon V' + \epsilon V \left(\sum_{j=0}^{2p} a_j \Delta x^{j+1} + O(\Delta x^{2(p+1)}) \right) - \epsilon^2 V \Delta x \frac{d^2 f(\tilde{u})}{4du^2}] d\xi = 0, \quad \forall\ V \in \mathcal{P}_p. \tag{41}$$

Truncating (9a) at the $O(\Delta x^{2(p+1)})$ term, substituting the result into (41), grouping terms having like powers of Δx, and following the reasoning of Theorem 1, we establish (38a). Consequently, (41) becomes

$$[Q_{p+1}V(1) - \int_{-1}^{1} Q_{p+1}V' d\xi]\Delta x^{p+1} + \sum_{k=p+2}^{2p+1} \left[Q_k(1)V(1) - \int_{-1}^{1} (Q_k V' + V \sum_{j=p+1}^{k-1} Q_j a_{k-1-j}) d\xi \right] \Delta x^k + O(\Delta x^{2(p+1)}) = 0. \tag{42a}$$

The ϵ^2 term in (41) is $O(\Delta x^{2(p+1)})$ and, thus, does not contribute to the leading terms of (42a). Equating coefficients of $O(\Delta x^k)$, $k = p+1, p+2, \ldots, 2p+1$, yields

$$Q_k(1)V(1) - \int_{-1}^{1} (Q_k V' + V \sum_{j=p+1}^{k-1} Q_j a_{k-1-j}) d\xi = 0, \qquad \forall\ V \in \mathcal{P}_p. \tag{42b}$$

Following the arguments of Theorem 1, an analysis of the $(k = p+1)$ leading term leads to (9c).

Once again, we employ an induction argument on (13) to complete the proof. Using (42b) with $V = 1$ and (13) yields

$$Q_k(1) = \int_{-1}^{1} \sum_{j=p+1}^{k-1} Q_j a_{k-1-j} d\xi = \int_{-1}^{1} \sum_{j=p+1}^{k-1} a_{k-1-j} \sum_{l=2p+1-j}^{j} c_{jl} P_l d\xi. \tag{43}$$

Using (8b) with (40b) yields (9d). Using (42b) with $V = \xi^i$ and (9d) leads to

$$i \int_{-1}^{1} Q_k(\xi)\xi^{i-1} d\xi = -\int_{-1}^{1} \xi^i \sum_{j=p+1}^{k-1} a_{k-1-j} \sum_{l=2p+1-j}^{j} c_{jl} P_l d\xi, \qquad i = 0, 1, \ldots, p. \tag{44}$$

Using (8b) yields

$$\int_{-1}^{1} Q_k(\xi)\xi^{i-1} d\xi = 0, \qquad i = 0, 1, \ldots 2p+1-k, \qquad k = p+2, p+3, \ldots, 2p+1, \tag{45}$$

which establishes (13). With all $O(\Delta x^{2p+1})$ terms vanishing, we conclude that (38b) is satisfied. □

A proof that these results apply to nonlinear systems of the form

$$\mathbf{u}' - \mathbf{f}(\mathbf{u}) = 0, \qquad x > 0, \qquad \mathbf{u}(0) = \mathbf{u}_0, \tag{46}$$

when $\mathbf{u}$ and $\mathbf{f}(\mathbf{u})$ are m-vectors and $\mathbf{f} \in C^{2p+1}$, follows the same lines. Likewise, an analysis of the global errors for nonlinear problems follows the linear analysis of Theorem 3.

3 Computational Results

We would like to demonstrate that the error estimation procedures based on superconvergence at the Radau points works well for conservation laws in regions of smooth solution. To do this, let us construct a (spatially discrete) discontinuous Galerkin form of (1) by multiplying it by a test function $\mathbf{v} \in \mathcal{L}_2(x_{n-1}, x_n)$, and integrating on (x_{n-1}, x_n) while integrating the flux term by parts. The result, upon a transformation to the canonical $(-1, 1)$ element using (7a), is [9, 11, 13]

$$\frac{\Delta x}{2} \frac{d}{dt} \int_{-1}^{1} \mathbf{v}^T \mathbf{u} d\xi + \mathbf{v}^T \mathbf{f}(\mathbf{u})|_{-1}^{1} - \int_{-1}^{1} \mathbf{v}_\xi^T \mathbf{f}(\mathbf{u}) d\xi = 0, \qquad \forall\ \mathbf{v} \in \mathcal{L}_2(-1, 1). \tag{47}$$

Continuing, we approximate $\mathbf{u}$ by $\mathbf{U}_n \in \mathcal{P}_p(x_{n-1}, x_n)$ in the form

$$\mathbf{U}_n(\xi, t) = \sum_{k=0}^{p} \mathbf{c}_{nk}(t) P_k(\xi), \tag{48}$$

test against functions $\mathbf{V}_n \in \mathcal{P}_p(x_{n-1}, x_n)$, and approximate the boundary flux $\mathbf{f}(\mathbf{U}_n(1,t))$ by a numerical flux $\mathbf{h}(\mathbf{U}_n(1,t), \mathbf{U}_{n+1}(-1,t))$ (with a similar expression approximating $\mathbf{f}(\mathbf{U}_n(-1,t))$). The resulting DGM is [9]

$$\frac{\Delta x}{2k+1}\dot{\mathbf{c}}_{nk} + \mathbf{h}(\mathbf{U}_n(1,t), \mathbf{U}_{n+1}(-1,t)) - (-1)^k \mathbf{h}(\mathbf{U}_{n-1}(1,t), \mathbf{U}_n(-1,t)) - \int_{-1}^{1} P_k'(\xi)\mathbf{f}(\mathbf{U}_n)d\xi = 0, \qquad k = 0, 1, \ldots, p, \qquad n = 1, 2, \ldots, N. \tag{49a}$$

Cockburn and Shu [13] presented results for several numerical flux functions. Herein, we use the local Lax-Friedrichs flux

$$\mathbf{h}(\mathbf{U}_L, \mathbf{U}_R) = \frac{1}{2}[\mathbf{f}(\mathbf{U}_L) + \mathbf{f}(\mathbf{U}_R) - \lambda_{max}(\mathbf{U}_R - \mathbf{U}_L)], \tag{49b}$$

where λ_{max} is the maximum absolute eigenvalue of the Jacobian $\mathbf{f}_{\mathbf{u}}(\mathbf{u})$, $\mathbf{u} \in [\mathbf{U}_L, \mathbf{U}_R]$. To complete the specification of the discrete problem, initial conditions are determined by interpolation at the roots of $R_{p+1}(\xi)$ on each element, the integral in (49a) is evaluated using eighth-order Gauss-Legendre quadrature, and temporal integration is performed by a forward Euler method when $p = 0$ and a fourth-order classical explicit Runge-Kutta method when $p > 0$. A time step Δt is chosen so that temporal errors are small relative to spatial errors consistent with the DGM "Courant condition" [13]

$$\max_{i\in[1,m],\ n\in[1,N]} |\lambda_{i,n}| \frac{\Delta t}{\Delta x} \le \frac{1}{2p+1}. \tag{50}$$

Here, $\lambda_{i,n}$, $i = 1, 2, \ldots, m$, are representative eigenvalues of $\mathbf{f}_{\mathbf{u}}(\mathbf{u})$ on (x_{n-1}, x_n), $n = 1, 2, \ldots, N$ and time t.

Following the results presented in §2, the error on (x_{n-1}, x_n) is approximated as

$$\mathbf{e}(\xi, t) \approx \mathbf{E}_n(\xi, t) = \boldsymbol{\alpha}_{p+1,n}(t) R_{p+1}(\xi). \tag{51}$$

The coefficient $\boldsymbol{\alpha}_{p+1,n}(t)$ is determined by replacing $\mathbf{u}$ in (47) by $\mathbf{U}_n + \mathbf{E}_n$, testing against $P_{p+1}(\xi)$, and following the solution procedure just described.

Example 1. The linear initial value problem

$$u_t + u_x = 0, \qquad t > 0, \qquad u(x, 0) = \sin \pi x, \tag{52a}$$

has the exact solution

$$u(x, t) = \sin \pi(x - t). \tag{52b}$$

Solving this problem on $-1 \le x < 1$, we present errors at $t = 1$ on an eight-element uniform mesh in Figure 1 for p ranging from 0 to 4. The Radau points of degree $p + 1$ are shown on each element as plus signs. Errors vanish at points close to the Radau points for all solutions except the one with $p = 0$, where there is no superconvergence for global errors and, thus, the error estimation procedure does not apply. As expected by the theory, superconvergence at Radau points becomes stronger as p increases with the roots of $e(x, t)$ getting closer to those of R_{p+1}.

To continue, we report the effectivity index

$$\theta = \frac{\|E\|_{\mathcal{L}_1}}{\|e\|_{\mathcal{L}_1}}, \tag{53}$$

in the $\mathcal{L}_1$ norm at $t = 1$ in Figure 2 as a function of the degrees of freedom $N(p+1)$ for $p = 1, 2, 3, 4$, and $N = 8, 16, \ldots, 256$. The effectivity indices appear to converge to unity as the mesh is refined. The apparent divergence with $p = 4$ is a result of the time error ceasing to be small relative to the very small spatial errors. The error estimates are robust with effectivity indices being in excess of 0.9 for all but the lowest degrees of freedom.

Example 2. We consider the initial value problem for the inviscid Burgers' equation

$$u_t + (\frac{u^2}{2})_x = 0, \qquad t > 0, \qquad u(x,0) = \frac{1 + \sin \pi x}{2} \tag{54}$$

on a $-1 \leq x < 1$ periodicity cell. The initial function steepens to form a shock at $t = 2/\pi$ which propagates in the positive x direction. The aim of this example is to examine the performance of error estimates in the presence of nonlinearity and discontinuity.

We solved problems with and without solution limiting. Solution limiting suppresses spurious oscillations near discontinuities when $p > 0$. For these tests, we used a moment limiting procedure of Biswas *et al.* [9]. Our initial results compare solutions with and without limiting at $t = 0.7$ with $p = 2$ (Figure 3). With no limiting, the solution features large "overshoots" near the discontinuity which are greatly reduced by the moment limiting. However, accuracy does not appear to be affected in regions where the solution is smooth and the results on error estimation that follow were obtained without limiting.

The global effectivity indices at $t = 0.4$ are shown as functions of degrees of freedom for $p = 0$ to 4 in Figure 4. The shock has not formed at $t = 0.4$; hence, effectivity indices are converging to unity for $p > 0$ as the degrees of freedom increase, albeit at a slower rate than for the linear problem of Example 1. This is due to the steepening of the wave as the shock forms. Rates would have improved if adaptive h-refinement were used in high-gradient regions. Effectivity indices do not converge to unity when shocks are present. They would if the region containing the shock were eliminated. To demonstrate this, we illustrate local effectivity indices (*i.e.*, the effectivity index on each element) in Figure 5. Results with limiting were virtually identical in regions removed from the shock. Large deviations from unity occur on elements near the shock (which is at $x = -0.65$ at $t = 0.7$). Other deviations occur with $p = 0$ because of the lack of superconvergence. Results with $p = 1$ show the effectivity index deviating from unity near $x = 0.4$. This is due to a low spatial error region near the inflection point in the wave. Temporal errors dominate in this region. The accuracy of the error estimates on each element clearly improves with both h- and p-refinement.

Example 3. We consider the nonlinear wave equation

$$u_{tt} - u_{xx} = u(2u^2 - 1) \tag{55a}$$

which can be written in the form (1) with the addition of a source term as

$$(u_1)_t + (u_1)_x = u_2, \qquad (u_2)_t - (u_2)_x = u_1(2u_1^2 - 1) \tag{55b}$$

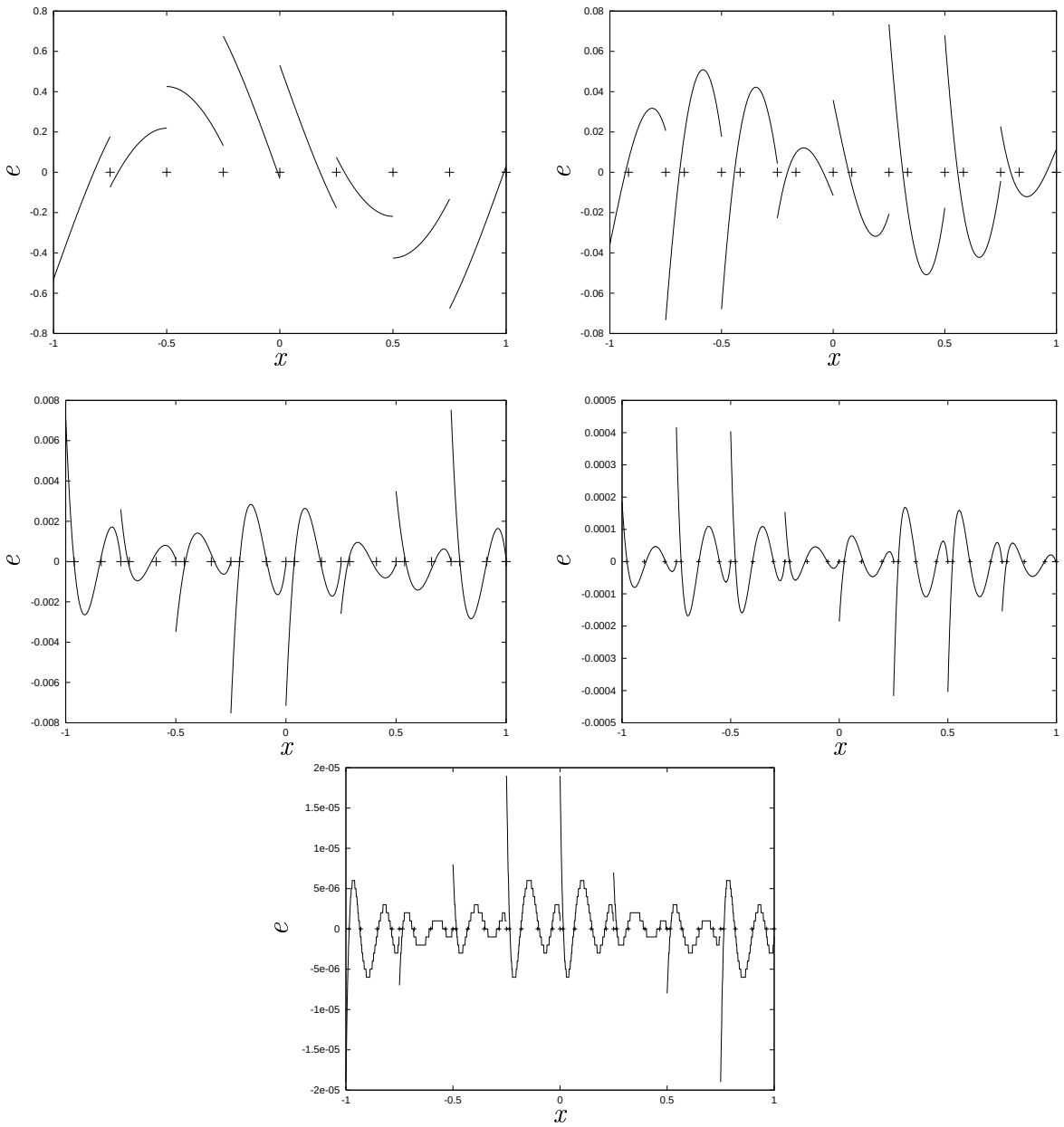

Figure 1: Discretization error of Example 1 at $t = 1$ using an eight-element mesh on $-1 \leq x < 1$ and $p = 0$ to 4 (upper left to lower center). Plus signs indicate the roots of R_{p+1}.

with $u_1 = u$. We choose the initial and boundary conditions such that the exact solution of (55a) is the solitary wave

$$u(x,t) = \operatorname{sech}(x \cosh \frac{1}{2} + t \sinh \frac{1}{2}). \tag{55c}$$

We solved problems without limiting using polynomials of degrees $p = 0$ to 4. The

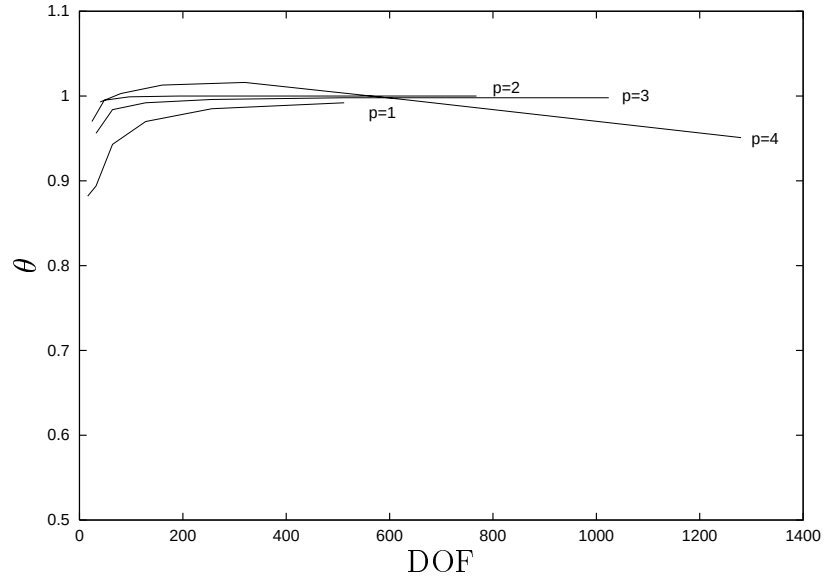

Figure 2: Effectivity indices as a function of degrees of freedom at $t = 1$ for Example 1 with $p = 1$ to 4.

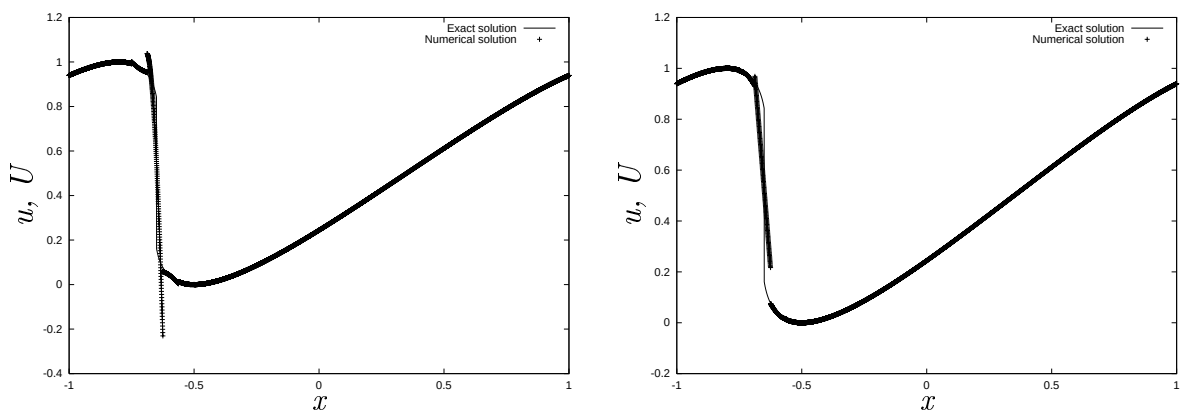

Figure 3: Exact and computed solutions of Example 2 at $t = 0.7$ without (left) and with (right) limiting for $p = 2$ and $N = 32$.

solution at $t = 1$ performed with $p = 2$ and $N = 64$ is shown on the left of Figure 6. Computations illustrating errors and effectivity indices were performed on the more restricted interval $-\pi/3 < x < \pi/3$. Effectivity indices for the error in $u_1 = u$ with p ranging from 1 to 4 and N ranging from 8 to 256 are shown as a function of the degrees of freedom on the right of Figure 6. Discretization errors and effectivity indices for $u_1 = u$ also appear in Table 1 to provide more detailed information. Left and right Radau polynomials are needed to estimate discretization errors of the two components of the solution of (55b). As noted, results present errors and effectivity indices for u_1 (the solution of (55a)). Results for u_2 would be similar.

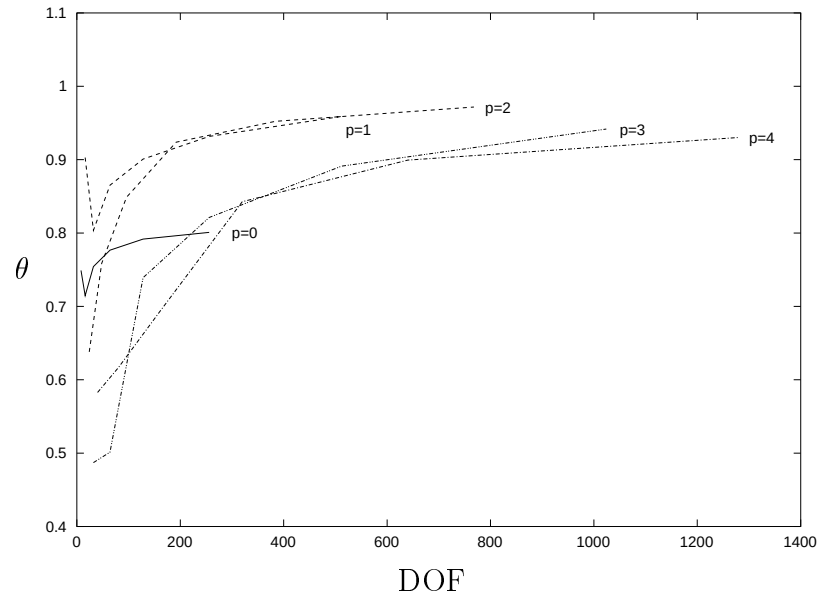

Figure 4: Effectivity indices as a function of degrees of freedom at $t = 0.4$ for Example 2 with $p = 0$ to 4.

Solutions of this nonlinear wave propagation problem appear to be converging at their expected $O(\Delta x^{p+1})$ rates in $\mathcal{L}_1$ (Table 1). Effectivity indices are within 0.5% of ideal for all combinations of mesh spacing and $p > 0$. Effectivity indices appear to converge to unity under both h- and p-refinement, for $p > 0$. Results with $p = 0$ are, as expected, inaccurate.

4 Conclusion

We have shown that the leading term of the local discretization error of a DGM with piecewise-polynomial approximations of degree p is proportional to a Radau polynomial of degree $p + 1$ on each element. We have also shown that the local discretization error is $O(\Delta x^{2(p+1)})$ at the downwind point and $O(\Delta x^{p+2})$ at the remaining roots of Radau polynomial of degree $p+1$ on each element. The high-order ($O(\Delta x^{2p+1})$) superconvergence of global errors at the downwind end of each element enables us to obtain similar results for the global discretization error. Convergence of local and global errors to the Radau polynomial of degree $p+1$ also occurs in p. This information establishes that *a posteriori* spatial error estimates of the DGM for hyperbolic conservation laws used by Biswas *et al.* [9] and Cockburn *et al.* [12] are asymptotically correct for smooth solutions under both mesh refinement and order enrichment.

In future investigations, we will attempt to construct error estimates that include the effects of temporal errors with the spatial error estimates provided here. Discretization error estimates in the presence of discontinuities would certainly be of interest. With this, we would also have to appraise the effect of limiting or other forms of stabilization on the error estimates. Error estimation for linear and nonlinear multi-dimensional problems

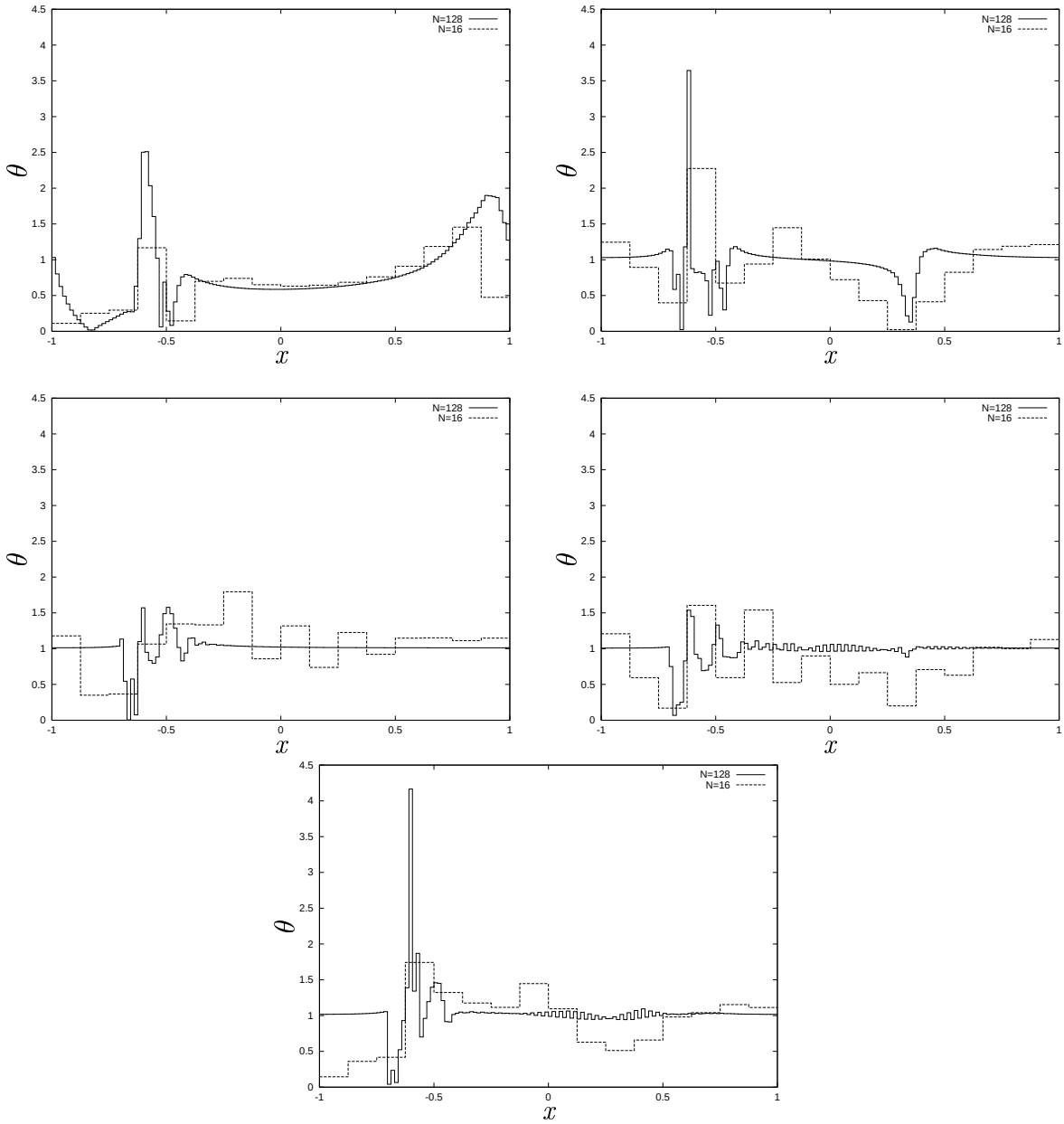

Figure 5: Local effectivity indices at $t = 0.7$ for Example 2 without limiting on meshes having $N = 16$ and 128 elements and $p = 0$ to 4 (upper left to lower right).

would be very important. Extending the present theory to tensor-product piecewise-polynomial spaces on quadrilaterals and hexahedra should be relatively straightforward. However, error analysis with non-product bases and on triangular and tetrahedral elements could be considerably more difficult. Radau points are not even defined in these situations. We also plan to investigate the use of these error estimation techniques when DGMs are applied to singularly perturbed convection-diffusion problems. Our earlier work [2] on two-point boundary value problems indicated that they could be very effec-

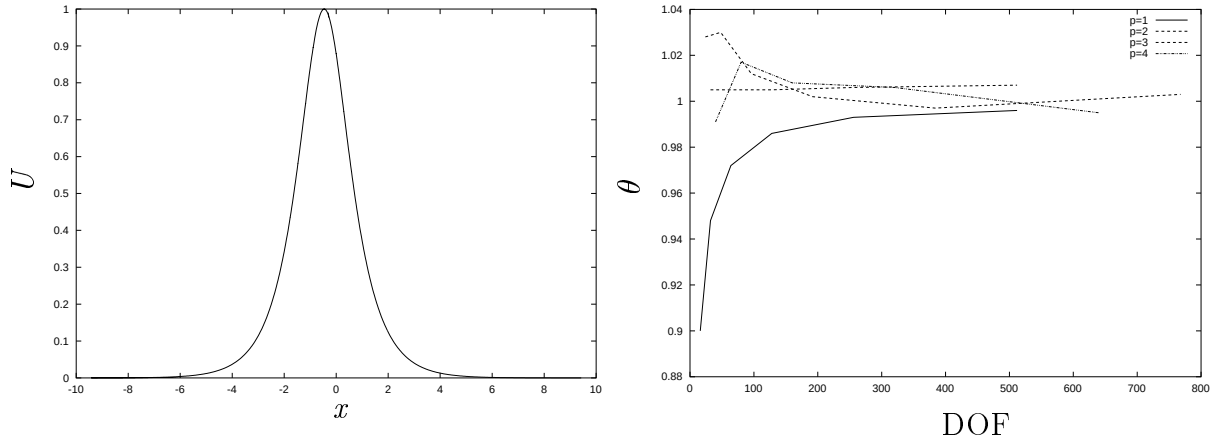

Figure 6: Solution of Example 3 at $t = 1$ on the interval $-3\pi < x < 3\pi$ (left). Global effectivity indices for the solution of (55a) at $t = 1$ as a function of degrees of freedom with p ranging from 1 to 4 (right).

tive.

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	$p=0$		$p=1$		$p=2$	
N	e	θ	e	θ	e	θ
8	2.16e-01	0.451	5.12e-03	0.900	1.88e-04	1.028
16	1.19e-01	0.411	1.19e-03	0.948	2.32e-05	1.030
32	6.39e-02	0.400	2.88e-04	0.972	2.90e-06	1.012
64	3.32e-02	0.395	7.06e-05	0.986	3.63e-07	1.002
128	1.69e-02	0.393	1.74e-05	0.993	4.53e-08	0.997
256	8.58e-03	0.393	4.34e-06	0.996	5.67e-09	1.003

	$p=3$		$p=4$	
N	e	θ	e	θ
8	7.12e-06	1.005	3.67e-07	0.991
16	4.38e-07	1.005	1.12e-08	1.017
32	2.70e-08	1.005	3.55e-10	1.008
64	1.68e-09	1.006	1.10e-11	1.006
128	1.04e-10	1.007	3.49e-13	0.995

Table 1: Discretization error and effectivity indices for the solution of (55a) at $t = 1$ as functions N and p.

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