ERROR ESTIMATION FOR DISCONTINUOUS GALERKIN SOLUTIONS OF MULTIDIMENSIONAL HYPERBOLIC PROBLEMS

Lilia Krivodonova and Joseph E. Flaherty
Scientific Computation Research Center
Rensselaer Polytechnic Institute
Troy, NY 12180-3590

Abstract

We analyze the discretization errors of discontinuous Galerkin solutions of steady two-dimensional hyperbolic conservation laws on unstructured meshes. We show that the leading term of the error on each element is a linear combination of orthogonal polynomials of degrees p and $p+1$. We further show that there is a strong superconvergence property at the outflow edge(s) of each element where the average discretization error converges as $O(h^{2p+1})$ compared to a global rate of $O(h^{p+1})$. Our analyses apply to both linear and nonlinear conservation laws with smooth solutions. We show how to use our theory to construct efficient and asymptotically exact *a posteriori* discretization error estimates and we apply these to some examples.

1 Introduction

The discontinuous Galerkin method (DGM), originally developed for neutron transport equations [16], is well suited to hyperbolic problems. The solution space is discontinuous relative to a structured or unstructured mesh and this allows solution discontinuities (*e.g.*, shocks) to be captured sharply relative to the computational mesh [7, 17, 18]. The DGM simplifies adaptivity since interelement continuity is neither required for h-refinement (mesh refinement and coarsening) nor p-refinement (method order variation) [17]. Physical quantities (*e.g.*, mass, momentum, and energy) are conserved element wise. The DGM has a communication pattern between elements sharing a common face, which simplifies parallel computation [11, 12].

A posteriori error estimates are traditionally used to guide adaptive enrichment by h- and p-refinement and to provide a measure of solution reliability. Somewhat naturally, progress in developing error estimation techniques has been slower for hyperbolic problems than for elliptic and parabolic equations [4, 6, 22, 23]. Nevertheless, some work exists. Süli [20] discusses *a posteriori* estimation for both linear and nonlinear problems; Houston *et al.* [13, 21] describe procedures for linear problems; Cockburn *et al.* [8, 9] consider nonlinear problems; and Pierce and Giles [15] and Larson and Barth [14] construct *a posteriori* error estimates for a linear functional of the solution, which is often more important than pointwise error estimates.

Adjerid *et al.* [3] proved that smooth DGM solutions of one-dimensional hyperbolic conservation laws using piecewise-polynomials of degree p have a higher $O(h^{p+2})$ rate of convergence (superconvergence) at the roots of the Radau polynomial of degree $p+1$ than they do elsewhere, which is generally $O(h^{p+1})$ in, *e.g.*, the $\mathcal{L}^2$ norm. They used this result to construct asymptotically correct estimates of the spatial discretization errors of DGM solutions of conservation laws. They further established a "strong" superconvergence at the "downwind" (outflow) end of every element where solutions converge at an $O(h^{2p+1})$, $p > 0$, rate.

It is natural to ask whether or not similar results are available for multi-dimensional problems. Adjerid [2] showed that Radau-based error estimates apply to two-dimensional problems with tensor-product bases. Herein, we show that very similar results apply to the DGM solution of conservation laws on unstructured triangular meshes. While Radau polynomials are not defined in higher than one dimension, we develop a theory in terms of a basis of orthogonal polynomials on triangular elements [17]. The key argument in the development is a demonstration that integrals of the discretization error vanish to leading order on each element and, simultaneously, on outflow edges of each element (§2). With this, an *a posteriori* estimate of the discretization error is obtained as a linear combination of orthogonal polynomials of degrees p and $p+1$ on each element (§3). Since Radau polynomials are a combination of (the orthogonal) Legendre polynomials of degrees p and $p+1$, our results may be regarded as their extension to two dimensions.

We demonstrate that a form of superconvergence exists by showing that the average local discretization error on outflow edge(s) of elements converges as $O(h^{2(p+1)})$. Thus, discretization errors propagate between elements at a high order, and we use this to obtain global discretization error estimates (§2). Results proved for linear problems extend to nonlinear problems provided that solutions remain smooth (§2). Examples indicate that our *a posteriori* error estimates are generally within 10% of the actual errors for wide ranges of mesh spacings and polynomial degrees.

2 Error Analysis

We consider the linear, steady, two-dimensional transport equation

$$\mathbf{a} \cdot \nabla u + cu = f(x, y), \qquad (x, y) \in \Omega, \tag{1}$$

subject to appropriate boundary conditions. We assume that $\mathbf{a} = [a_1, a_2]^T$ and c are constant and $f \in \mathcal{L}^2(\Omega)$, where $\Omega \subset \Re^2$. We discretize Ω into a collection of N triangular elements Ω_j, $j = 1, 2, \ldots, N$, and assume, for simplicity, that this can be done without error. Additional mesh regularity conditions will be subsequently stated. We approximate $u(x, y)$ by a piecewise polynomial function $U(x, y)$ whose restriction $U_j(x, y)$ to Ω_j is an element of $\mathcal{P}_p(\Omega_j)$ consisting of complete polynomials of degree p in two dimensions.

With this, the discontinuous Galerkin form of (1) may be written as [19]

$$\int_{\partial\Omega_j^-} V\mathbf{a}\cdot\mathbf{n}(U^- - U_j)\,dt + \iint_{\Omega_j} V(\mathbf{a}\cdot\nabla U_j + cU_j)\,ds = \iint_{\Omega_j} Vf\,ds, \qquad \forall V \in \mathcal{P}_p(\Omega_j). \quad (2)$$

The boundary $\partial\Omega_j$ of Ω_j has unit outward normal vector $\mathbf{n}$ and is divided into segments $\partial\Omega_j^-$ and $\partial\Omega_j^+$ where the flow vector $\mathbf{a}$ is, respectively, into and out of the element. In a similar manner, U^- and U^+, respectively, represent solution values entering and leaving Ω_j.

Let us begin with the local error

$$\epsilon = u - U_j, \quad (3)$$

where there is no error entering Ω_j; thus, $U_j^-|_{\partial\Omega^-} = u$. A continuous exact solution satisfies (2) without the jump term across $\partial\Omega_j^-$, *i.e.*,

$$\iint_{\Omega_j} V(\mathbf{a}\cdot\nabla u + cu)\,ds = \iint_{\Omega_j} Vf\,ds, \qquad \forall V \in \mathcal{P}_p. \quad (4)$$

Subtracting (2) from (4) and using the divergence theorem results in the *Galerkin orthogonality* condition

$$\int_{\partial\Omega_j^+} \mathbf{a}\cdot\mathbf{n}\epsilon V\,dt - \iint_{\Omega_j} \epsilon(\mathbf{a}\cdot\nabla V - cV)\,ds = 0, \qquad \forall V \in \mathcal{P}_p. \quad (5)$$

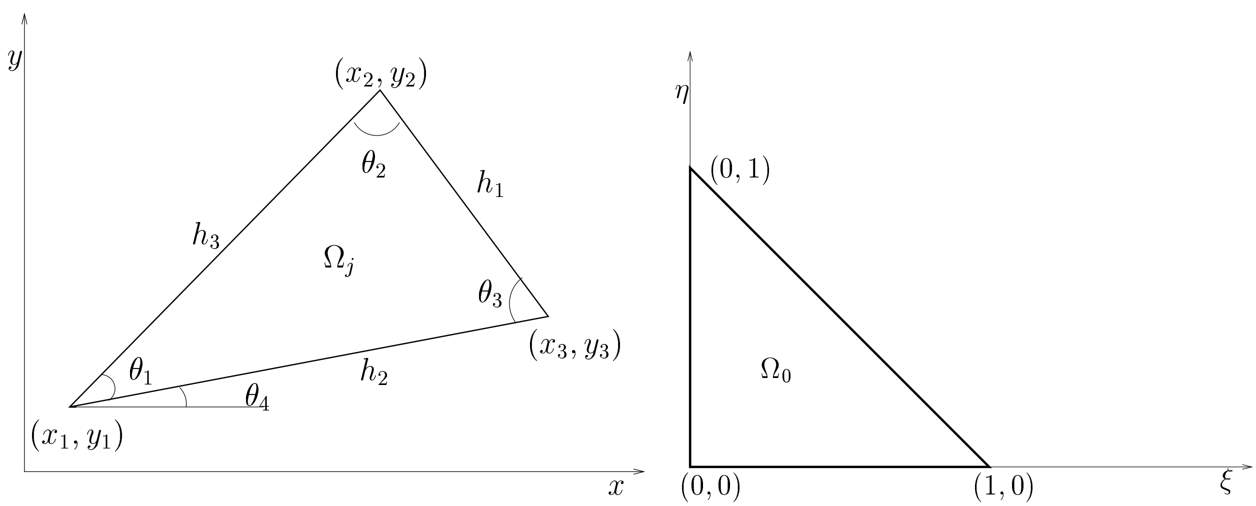

Figure 1: Mapping of a triangle Ω_j (left) onto a canonical triangle Ω_0 (right).

For convenience, we map element Ω_j onto a canonical right triangle $\Omega_0 = \{(\xi,\eta)|0 \le \eta \le 1-\xi,\ 0 \le \xi \le 1\}$ (Figure 1) using a linear transformation [5]. The Jacobian of this transformation is

$$\det(\mathbf{J}_j) = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} = h_2 h_3 \sin\theta_1, \quad (6)$$

where h_i, $i = 2, 3$, are two longest edges of Ω_j and θ_1 is the angle between them (Figure 1). After transformation, (5) has the scalar form

$$\begin{aligned}\int_{\partial\Omega_0^+} \mathbf{a}\cdot\mathbf{n}\epsilon V \det(\mathbf{J}_j)\, dt - \iint_{\Omega_0} \epsilon[(a_1\xi_x + a_2\xi_y)V_\xi \quad &+ \quad (a_1\eta_x + a_2\eta_y)V_\eta - cV] \det(\mathbf{J}_j)\, d\xi d\eta \\ &= \quad 0, \qquad \forall V \in \mathcal{P}_p(\Omega_0), \end{aligned} \tag{7}$$

with the understanding that $\epsilon(\xi, \eta) = \epsilon(x(\xi, \eta), y(\xi, \eta))$.

Lemma 1. *Let the mesh be quasi-uniform, i.e., there exist constants k_i, $i = 1, 2, 3$, such that $h = k_i h_i$, $i = 1, 2, 3$, for all $h \le \bar{h}$, where h is the longest element edge in the mesh. Then, there exists a vector $\boldsymbol{\alpha} = [\alpha_1, \alpha_2]^T$ and scalars β and γ such that*

$$\begin{aligned} a_1\xi_x + a_2\xi_y = \alpha_1\frac{h}{\det(\mathbf{J}_j)}, &\qquad a_1\eta_x + a_2\eta_y = \alpha_2\frac{h}{\det(\mathbf{J}_j)}, \\ \det(\mathbf{J}_j)|_{\partial\Omega_0} = \beta h, &\qquad c\det(\mathbf{J}_j) = \gamma h^2. \end{aligned} \tag{8}$$

Proof. The linear mapping (6), quasi-uniformity condition, and local geometry (Figure 1) imply

$$a_1\xi_x + a_2\xi_y = \frac{a_1(y_3 - y_1) - a_2(x_3 - x_1)}{\det(\mathbf{J}_j)} = \frac{h_2(a_1\sin\theta_4 - a_2\cos\theta_4)}{\det(\mathbf{J}_j)}, \tag{9}$$

which leads to the first of (8). The remaining relations follow in similar fashion. ☐

Applying (8) to (7) yields the scaled equation

$$\int_{\partial\Omega_0^+} \beta\mathbf{a}\cdot\mathbf{n}\epsilon V\, dt - \iint_{\Omega_0} \epsilon(\boldsymbol{\alpha}\cdot\nabla V - \gamma V h)\, ds = 0, \qquad \forall V \in \mathcal{P}_p(\Omega_0). \tag{10}$$

We construct a basis for $\mathcal{P}_p(\Omega_0)$ as a set of polynomials that are orthonormal in $\mathcal{L}_2(\Omega_0)$ [10, 17].

$$\begin{aligned}\int_0^1\int_0^{1-\xi} \varphi_i^k\varphi_m^n\, d\xi d\eta = \delta_{i,m}\delta_{k,n}, \qquad i \quad &= \quad 1, 2, \ldots, k+1, \qquad m = 1, 2, \ldots, n+1, \\ k, n \quad &= \quad 0, 1, \ldots, p, \end{aligned} \tag{11}$$

where superscripts indicate the polynomial degree and subscripts the basis elements of a given degree. With this basis, we obtain the following expression for the local error.

Theorem 1. *Let $u \in C^{2p+3}(\Omega_j)$ be a solution of (1) and $U_j \in \mathcal{P}_p(\Omega_j)$ be a solution of (2) with $U^-|_{\partial\Omega_j^-} = u$. Then*

$$\epsilon(\xi, \eta) = \sum_{k=p+1}^{2(p+1)} Q_k(\xi, \eta)h^k + O(h^{2p+3}) \tag{12}$$

with

$$\iint_{\Omega_0} Q_{p+1} V \, d\xi d\eta = 0, \qquad \forall V \in \mathcal{P}_{p-1}, \tag{13a}$$

$$\int_{\partial\Omega_0^+} \beta \mathbf{a} \cdot \mathbf{n} Q_{p+1} V \, dt = 0, \qquad \forall V \in \mathcal{P}_p, \tag{13b}$$

$$\int_{\partial\Omega_0^+} \beta \mathbf{a} \cdot \mathbf{n} Q_k \, dt = 0, \qquad p+1 \leq k \leq 2p+1, \tag{13c}$$

$$Q_{p+1} = \sum_{k=p}^{p+1} \sum_{i=1}^{k+1} c_i^k \varphi_i^k. \tag{13d}$$

Proof. The smoothness of u allows us to expand the error in a series of the form (12) with a lower limit of $k = 0$. However, using reasoning similar to that which follows, we easily show that $Q_k \equiv 0, k = 0, 1, \ldots, p$. Substituting (12) into (10) and grouping terms having like powers of h yields

$$\left[\int_{\partial\Omega_0^+} \beta \mathbf{a} \cdot \mathbf{n} Q_{p+1} V \, dt - \iint_{\Omega_0} Q_{p+1} \boldsymbol{\alpha} \cdot \nabla V \, ds \right] h^{p+1} +$$
$$\sum_{k=p+2}^{2p+2} \left[\int_{\partial\Omega_0^+} \beta \mathbf{a} \cdot \mathbf{n} Q_k V \, dt - \iint_{\Omega_0} (Q_k \boldsymbol{\alpha} \cdot \nabla V - \gamma Q_{k-1} V) \, ds \right] h^k + O(h^{2p+3}) = 0,$$
$$\forall V \in \mathcal{P}_p(\Omega_0). \tag{14}$$

The leading term of (14) satisfies

$$\int_{\partial\Omega_0^+} \beta \mathbf{a} \cdot \mathbf{n} Q_{p+1} V \, dt - \iint_{\Omega_0} Q_{p+1} \boldsymbol{\alpha} \cdot \nabla V \, ds = 0, \qquad \forall V \in \mathcal{P}_p(\Omega_0). \tag{15}$$

With a uniform flow $\boldsymbol{\alpha}$, we need only distinguish cases where Ω_j has (i) one or (ii) two outflow boundaries. We consider the case of one outflow edge first and suppose that the mapping of Ω_j onto Ω_0 places this edge on $\eta = 0$ (Figure 1). Then (15) becomes

$$\int_0^1 \beta \mathbf{a} \cdot \mathbf{n} Q_{p+1}(\xi, 0) V(\xi, 0) \, d\xi - \iint_{\Omega_0} Q_{p+1} \boldsymbol{\alpha} \cdot \nabla V \, d\xi d\eta = 0, \qquad \forall V \in \mathcal{P}_p. \tag{16}$$

Fixing q on $0 \leq q \leq p$, we consider a space Π_q, consisting of all monomials of degree q, *i.e.*,

$$\Pi_q = \text{span}\{\xi^q, \xi^{q-1}\eta, \ldots, \eta^q\}. \tag{17}$$

Since $\mathcal{P}_p = \Pi_0 \oplus \Pi_1 \oplus \cdots \oplus \Pi_p$, we may establish (13a,b) by proving

$$\iint\limits_{\Omega_0} Q_{p+1} V \, d\xi d\eta = 0, \qquad \forall V \in \Pi_{q-1}, \qquad 0 \le q \le p, \tag{18a}$$

and

$$\int_0^1 Q_{p+1}(\xi, 0) V(\xi, 0) \, d\xi = 0, \qquad \forall V \in \Pi_q, \qquad 0 \le q \le p. \tag{18b}$$

Setting $V = 1$ in (16) yields

$$\int_0^1 \beta \mathbf{a} \cdot \mathbf{n} Q_{p+1}(\xi, 0) \, d\xi = 0, \tag{19}$$

while satisfying (18b) trivially.

For any $q > 0$, select $V = \xi^{q-i}\eta^i$, $i = 1, 2, \ldots, q$, and substitute into (16) to obtain

$$\iint\limits_{\Omega_0} Q_{p+1} \boldsymbol{\alpha} \cdot \nabla(\xi^{q-i}\eta^i) \, d\xi d\eta = 0, \qquad i = 1, 2, \ldots, q. \tag{20}$$

The line integral in (16) has vanished since each V vanishes on $\eta = 0$. Demonstrating that (20) is satisfied for $V = \xi^{q-i}\eta^i$, $i = 1, 2, \ldots, q$, $q > 0$, is equivalent to satisfying (18a) for $\forall V \in \Pi_{q-1}$, since the gradient operator involves a differentiation. Thus, we have established (18a). Testing against $V = \xi^q$ is redundant in the sense that $\boldsymbol{\alpha} \cdot \nabla(\xi^{q-i}\eta^i)$, $i = 1, 2, \ldots, q$, form a basis for Π_{q-1}, and $\boldsymbol{\alpha} \cdot \nabla \xi^q$ can be expressed in terms of this basis. Thus, (20) is also satisfied when $i = 0$.

Using (18a) in (16) yields (18b). Combining results for all $q = 0, 1, \ldots, p$ proves (13a,b). The analysis with two outflow edges follows the same logic.

Due to orthogonality of φ_i^k, (13d) follows directly from (13a). We establish (13c) by using an induction argument. In particular, we seek to prove

$$Q_k = \sum_{l=2(p+1)-(k+1)}^{k} \sum_{i=1}^{l+1} c_i^l \varphi_i^l, \qquad k = p+1, p+2, \ldots, 2p+1, \tag{21}$$

with (13d) serving as the basis of the induction.

The $O(h^k)$ term in (14) satisfies

$$\int\limits_{\partial\Omega_0^+} \beta \mathbf{a} \cdot \mathbf{n} Q_k V \, dt - \iint\limits_{\Omega_0} (Q_k \boldsymbol{\alpha} \cdot \nabla V - \gamma Q_{k-1} V) \, ds = 0, \qquad \forall V \in \mathcal{P}_p(\Omega_0). \tag{22}$$

Testing against $V \in \Pi_i$, $i = 0, 1, \ldots, 2(p+1) - (k+1)$, and using (11) and (21) leads to

$$\int\limits_{\partial\Omega_0^+} \beta \mathbf{a} \cdot \mathbf{n} Q_k V \, dt - \iint\limits_{\Omega_0} (Q_k \boldsymbol{\alpha} \cdot \nabla V) \, ds = 0, \qquad \forall V \in \Pi_i, \qquad i = 0, 1, \ldots, 2(p+1) - (k+1). \tag{23}$$

Arguments similar to those used earlier in the theorem show that the line and area integrals vanish simultaneously. This, combined with (11), completes the induction argument and establishes (21). Choosing $V = 1$ in (22) yields

$$\int_{\partial\Omega_0^+} \beta \mathbf{a}\cdot\mathbf{n} Q_k V \, dt = -\iint_{\Omega_0} \gamma Q_{k-1} \, ds, \qquad k = p+2, p+3, \ldots, 2(p+1). \tag{24}$$

A final use of (21) with (11) shows that the right side of (24) vanishes for $k = p+2, p+3, \ldots, 2p+1$, and, hence, yields (13c). □

The results (12, 13) are similar to those of Adjerid *et al.* [2] in one dimension. Thus, the leading term of the local discretization error (13d) is a linear combination of orthogonal polynomials of degrees p and $p+1$. There is also a strong superconvergence (13c), at least on the average, at the outflow edge(s) of each element. In particular, integrating (12) on Ω_j^+ while using (13c) yields

$$\int_{\partial\Omega_j^+} \mathbf{a}\cdot\mathbf{n}\epsilon \, dt = O(h^{2p+3}). \tag{25}$$

The following corollary provides yet more specific information.

Corollary 1. *Under the conditions of Theorem 1*

$$Q_k(\xi(\tau), \eta(\tau)) = \sum_{i=2(p+1)-k}^{k} d_i P_i(\tau), \qquad k = p+1, p+2, \ldots, 2p+2, \qquad (\xi, \eta) \in \Omega_0^+, \tag{26}$$

where $P_k(\tau)$ is the Legendre polynomial of degree k in $\tau \in [-1, 1]$.

Proof. Assuming Q_k, $k = p+1, p+2, \ldots, 2p+1$, does not vanish identically on Ω_0^+, we parameterize (13b) by τ to obtain

$$\int_{-1}^{1} Q_{p+1}(\tau) V(\tau) \, d\tau = 0, \qquad \forall V \in \mathcal{P}_p. \tag{27}$$

Expanding $Q_{p+1}(\tau)$ and $V(\tau)$ in series of Legendre polynomials and using their orthogonality properties [1] yields

$$Q_{p+1}(\xi(\tau), \eta(\tau)) = d_{p+1} P_{p+1}(\tau), \tag{28}$$

which proves (26) with $k = p+1$. Using this as a base, an induction argument on k in (23) completes the proof. □

Superconvergence at the Legendre points on the outflow boundary, although interesting, does not resolve the issue of superconvergence for multi-dimensional problems on unstructured meshes. We would normally expect the existence of curves emanating from these points where the solution converges more rapidly than elsewhere. Numerical computations reveal such curves; however, they appear to be problem dependent and not identifiable in a general sense. Thus, the only superconvergence that we are able to report at this time is that which occurs in the average sense of (13c).

The local error estimates of Theorem 1 and the strong superconvergence (13c) at outflow edges enable us to obtain similar results for the global discretization error

$$e_j = u - U_j, \qquad (x,y) \in \Omega_j. \tag{29}$$

Let us begin with a lemma that characterizes the propagation of errors from outflow to inflow boundaries.

Lemma 2. *Let Ω_{m-1} be a neighboring element (or elements) sharing the inflow boundary $\partial\Omega_m^-$ of Ω_m. Let the error e_{m-1} on Ω_{m-1} satisfy (12, 13a,b) and*

$$Q_{k,m-1} = \sum_{l=2(p+1)-(k+1)}^{k} \sum_{i=1}^{l+1} c_{i,m-1}^l \varphi_i^l, \qquad k = p+1, p+2, \ldots, q, \tag{30}$$

for some q such that $p+1 < q \leq 2p+1$. If u satisfies the conditions of Theorem 1, then

$$e_m(\xi,\eta) = \sum_{k=p+1}^{q+1} Q_{k,m}(\xi,\eta)h^k + O(h^{q+2}), \tag{31}$$

$$\iint\limits_{\Omega_{0,m}} Q_{p+1,m} V \, d\xi d\eta = 0, \qquad \forall V \in \mathcal{P}_{p-1}, \tag{32a}$$

$$\int\limits_{\partial\Omega_{0,m}^+} \beta_m \mathbf{a}\cdot\mathbf{n} Q_{p+1,m} V \, dt = 0, \qquad \forall V \in \mathcal{P}_p, \tag{32b}$$

$$\int\limits_{\partial\Omega_{0,m}^+} \beta_m \mathbf{a}\cdot\mathbf{n} Q_{k,m} \, dt = 0, \qquad p+1 \leq k \leq q, \tag{32c}$$

$$\int\limits_{\partial\Omega_{0,m}^+} \beta_m \mathbf{a}\cdot\mathbf{n} Q_{q+1,m} \, dt = -\int\limits_{\partial\Omega_{0,m-1}^+} \beta_{m-1} \mathbf{a}\cdot\mathbf{n} Q_{q+1,m-1} \, dt - \iint\limits_{\Omega_{0,m}} \gamma_m Q_{q,m} \, ds. \tag{33}$$

Proof. Subtracting (4) from (2) and using (29) yields

$$\int\limits_{\partial\Omega_m^-} \mathbf{a}\cdot\mathbf{n}(U_m - U^-)V \, dt + \iint\limits_{\Omega_m} V(\mathbf{a}\cdot\nabla e_m + c e_m) \, ds = 0, \qquad \forall V \in \mathcal{P}_p. \tag{34}$$

With the present notation, $U^- = U_{m-1}$ and $\partial\Omega_m^- = \partial\Omega_{m-1}^+$ (with a change in the direction of the normal vector and integration path). Subscripts have been added to all terms and coefficients to indicate elements they belong to. Then, adding and subtracting the exact solution u to the line integral while applying the divergence theorem to the area integral yields

$$\int_{\partial\Omega_m^+} \mathbf{a}\cdot\mathbf{n} e_m V\,dt + \int_{\partial\Omega_{m-1}^+} \mathbf{a}\cdot\mathbf{n} e_{m-1} V\,dt - \iint_{\Omega_m} e_m(\mathbf{a}\cdot\nabla V - cV)\,ds = 0, \qquad \forall V \in \mathcal{P}_p. \quad (35)$$

Mapping (35) to the canonical element, using (31), and collecting terms of like powers of h leads to

$$\left[\int_{\partial\Omega_{0,m-1}^+} \beta_{m-1}\mathbf{a}\cdot\mathbf{n} Q_{p+1,m-1} V\,d\tau + \int_{\partial\Omega_{0,m}^+} \beta_m \mathbf{a}\cdot\mathbf{n} Q_{p+1,m} V\,dt - \iint_{\Omega_{0,m}} Q_{p+1,m}\boldsymbol{\alpha}_m\cdot\nabla V\,ds\right] h^{p+1} + \sum_{k=p+2}^{q+1}\left[\int_{\partial\Omega_{0,m-1}^+} \beta_{m-1}\mathbf{a}\cdot\mathbf{n} Q_{k,m-1} V\,dt + \int_{\partial\Omega_{0,m}^+} \beta_m\mathbf{a}\cdot\mathbf{n} Q_{k,m} V\,dt - \iint_{\Omega_{0,m}} (Q_{k,m}\boldsymbol{\alpha}_m\cdot\nabla V - \gamma_m Q_{k-1,m} V)\,ds\right] h^k + O(h^{q+2}) = 0,$$

$$\forall V \in \mathcal{P}_p. \quad (36)$$

The leading terms of (14) and (36) coincide since $Q_{p+1,m-1}$ is assumed to satisfy (13b). Thus, establishing (32a,b) may be done by using the arguments of Theorem 1. To complete the proof we use an induction argument on k in (36) to show

$$Q_{k,m} = \sum_{l=2(p+1)-(k+1)}^{k} \sum_{i=1}^{l+1} c_{i,m}^l \varphi_i^l, \qquad k = p+1, p+2, \ldots, q, \quad (37)$$

with (32a) serving as the base of the induction. Testing against $V \in \Pi_i$, $i = 0, 1, \ldots, q-k$, and using (11), (30), and (37) leads to

$$\int_{\partial\Omega_{0,m}^+} \beta_m\mathbf{a}\cdot\mathbf{n} Q_{k,m} V\,dt - \iint_{\Omega_{0,m}} (Q_{k,m}\boldsymbol{\alpha}_m\cdot\nabla V)\,ds = 0, \qquad \forall V \in \Pi_i, \qquad i = 0, 1, \ldots, q-k. \quad (38)$$

Equation (38) coincides with (23) for $i \le q$. Thus, the propagated error e_{m-1} does not contribute to $Q_{k,m}$, $k = p+1, p+2, \ldots, q$, and the analysis follows the lines of Theorem 1 and proves (32c). The $O(h^{q+1})$ terms in (36) yield

$$\int_{\partial\Omega_{0,m-1}^+} \beta_{m-1}\mathbf{a}\cdot\mathbf{n} Q_{q+1,m-1} V\,dt + \int_{\partial\Omega_{0,m}^+} \beta_m\mathbf{a}\cdot\mathbf{n} Q_{q+1,m} V\,dt - \iint_{\Omega_{0,m}} (Q_{q+1,m}\boldsymbol{\alpha}_m\cdot\nabla V - \gamma_m Q_{q,m} V)\,ds = 0. \quad (39)$$

Selecting $V = 1$ yields (33). ☐

We have shown (33) that the error propagation from one element to the next is $O(h^{2p+3})$ and this is dominated by the local discretization error. This allows us to obtain a global discretization error estimate.

Theorem 2. *Let u satisfy the conditions of Theorem 1, then the global discretization error (29) satisfies (31), (32a,b), and*

$$\int_{\partial\Omega^+_{0,j}} \beta_j \mathbf{a}\cdot\mathbf{n} Q_{k,j}\, dt = 0, \qquad p+1 \leq k \leq 2p. \tag{40}$$

Proof. The proof consists of two steps: first, we apply Lemma 2 consequently to find the propagated error; then, we use it again on Ω_j and the neighboring element(s) Ω_{j-1} to obtain the global error. Consider (35) and, following the characteristics, construct a path through elements Ω_i, $i = 1, 2, \ldots, j-1$, from Ω_j to the boundary [19]. Apply Lemma 2 recursively to Ω_i to show that (32a,b) are satisfied on Ω_i, $i = 1, 2, \ldots, j-1$. Use (33) with a change to the physical domain to obtain

$$\int_{\partial\Omega^+_{j-1}} \mathbf{a}\cdot\mathbf{n} Q_{2p+2,j-1}\, dt = \sum_{i=1}^{j-1} \iint_{\Omega_i} (-1)^{j-i} c Q_{2p+1,i}\, ds = O(h). \tag{41}$$

Thus, there is a loss of one order at the outflow boundary due to summing over $j-1 = O(1/h)$ elements. The assumption (30) is satisfied with $q = 2p$. Applying Lemma 2 to Ω_{j-1} and Ω_j proves (32a,b) on Ω_j and (40). □

Remark. It follows from (40) that

$$\int_{\partial\Omega^+_j} \mathbf{a}\cdot\mathbf{n} e_j\, dt = O(h^{2(p+1)}). \tag{42}$$

Finally, the results of Theorems 1 and 2 extend to nonlinear problems of the form

$$\nabla \cdot F(u) = 0, \qquad (x, y) \in \Omega, \tag{43}$$

provided that the solution is smooth.

Theorem 3. *Let $u \in C^{2p+3}(\Omega)$ and $U_j \in \mathcal{P}_p(\Omega_j)$ be, respectively, exact and discontinuous Galerkin solutions of (43). If $F(u) \in C^{2p+1}$, then the global discretization error satisfies (31, 32a,b,40).*

Proof. The assumed smoothness of F allows us to linearize the problem about a convenient solution and follow the arguments of Theorems 1 and 2. □

3 Computational Results

The analysis of §2 indicates that the discretization error may be approximated by the leading term of the series (31). Thus, using (13d), we have

$$e_j \approx Q_{p+1,j}(\xi,\eta) = \sum_{i=1}^{p+2} c_{i,j}^{p+1}\varphi_i^{p+1}(\xi,\eta) + \sum_{i=1}^{p+1} c_{i,j}^{p}\varphi_i^{p}(\xi,\eta). \tag{44}$$

The $2p+3$ coefficients $c_{i,j}^p$, $i = 1,2,\ldots,p+1$, $c_{i,j}^{p+1}$, $i = 1,2,\ldots,p+2$, may be determined on Ω_j by (i) satisfying the DGM (2) with U_j replaced by $U_j + Q_{p+1,j}$ and (ii) satisfying the outflow condition (32b). Thus, transforming (2) to Ω_0 using the linear mapping with Jacobian (6), using the divergence theorem, and replacing U_j by $U_j+Q_{p+1,j}$, we have

$$\int_{\partial\Omega_{0,j}^-} \beta_j \mathbf{a}\cdot\mathbf{n}(U_{j-1} + Q_{p+1,j-1} - U_j - Q_{p+1,j})V\det(\mathbf{J}_j)\,dt - \iint_{\Omega_{0,j}} (U_j + Q_{p+1,j})(\boldsymbol{\alpha}_j\cdot\nabla V$$

$$- \gamma_j V)\det(\mathbf{J}_j)\,ds = \iint_{\Omega_{0,j}} fV\det(\mathbf{J}_j)\,ds, \qquad \forall V \in \{\varphi_i^{p+1}\}_{i=1}^{p+2}, \tag{45a}$$

where element $j-1$ is a neighbor (or neighbors) of element j. Condition (32b) yields

$$\int_{\partial\Omega_{0,j}^+} \beta_j \mathbf{a}\cdot\mathbf{n}Q_{p+1,j}V\,dt = 0, \qquad \forall V \in \{\varphi_i^{p}\}_{i=1}^{p+1}. \tag{45b}$$

After evaluation of the inner products, the $(2p+3)\times(2p+3)$ linear system (45) may be solved for $c_{i,j}^p$, $i = 1,2,\ldots,p+1$, $c_{i,j}^{p+1}$, $i = 1,2,\ldots,p+2$, by, *e.g.*, Gaussian elimination. This is computationally inexpensive for practical values of p. For linear problems and linear mappings (6), the components of the "stiffness matrices" resulting from (45) may be computed once and used for all elements after an adjustment by the elemental Jacobians of the coordinate transformation (6). For nonlinear problems (43), the stiffness matrices would have to be generated and the resulting linear system solved for each element.

We demonstrate the effectiveness of the error estimation procedure (44, 45) by solving two nonlinear problems.

Example 1. Consider the quasi-linear equation

$$2u_x + u_y = \frac{3}{2u} \tag{46a}$$

on the unit square $0 \le x,y \le 1$, with Dirichlet boundary conditions chosen such that the exact solution is

$$u = \sqrt{x+y+1}. \tag{46b}$$

We solved (46) using software for transient problems [17] and integrating to a steady state. Computations involved sequences of unstructured meshes with the initial coarse mesh containing $N = 16$ triangular elements and polynomials of degrees $p = 0$ to 4. Let E be the piecewise-polynomial error estimate that takes on the value $Q_{p+1,j}$ on Ω_j, $j = 1, 2, \ldots, N$, and let

$$\Theta = \frac{\|E\|_1}{\|e\|_1} \tag{47}$$

be the effectivity index in the $\mathcal{L}_1$ norm. We present values of the discretization errors and effectivity indices for (46) in Table 1. Error estimates are within 10% of exact for all mesh spacings and polynomial degrees. Errors appear to converge at their expected $O(h^{p+1})$ rates in $\mathcal{L}_1$ for this problem with a smooth solution. The high accuracy of the computed solution and the uncertainty of reaching an exact steady state make it difficult to determine if the effectivity indices are converging to unity as either $h \to 0$ or $p \to \infty$.

N	16		56		160	
p	e	Θ	e	Θ	e	Θ
0	4.85e-02	1.0116	2.49e-02	1.0304	1.49e-02	1.0418
1	8.27e-03	1.0022	2.16e-04	1.0537	7.85e-05	1.0266
2	3.11e-05	0.9609	4.16e-06	0.9267	9.24e-07	0.9054
3	1.71e-06	1.0161	1.04e-07	1.0546	1.47e-08	1.0054
4	1.07e-07	1.0597	3.32e-09	1.0097	2.8e-10	0.9203

Table 1: Discretization errors and effectivity indices for Example 1 for a range of mesh spacings and polynomial degrees.

We also report the rate of convergence on the outflow boundary by conducting a sequence of computations on meshes obtained by dividing Ω into uniform square elements and subsequently dividing these into right triangles by connecting their upper left and lower right vertices. Let

$$I^+ = \int_L (u - U^-)\,dt, \tag{48}$$

where L is the line connecting the upper left corner of the domain $(0, 1)$ to the lower right corner $(1, 0)$. Here, U^- is the numerical solution on the left of L; hence, we may interpret I^+ as the average error on outflow boundaries of those elements to the left of L. To compare, we also compute

$$I^- = \int_L (u - U^+)\,dt, \tag{49}$$

where U^+ is the numerical solution on the right of L. Thus, I^- may be interpreted as the average error on inflow boundaries of those elements to the right of L.

Results for $p = 0, 1, 2, 3$, and $N = 8, 32, 128$, are presented in Table 2. Estimates of the convergence rates compare well with the theoretical $O(h^{2p+1})$ and $O(h^{p+1})$ rates on outflow and inflow boundaries, respectively. The average numerical solution is, thus, considerably more accurate along an outflow edge than it is along an inflow edge, even for very coarse meshes.

	$p=0$				$p=1$			
N	I^-	r	I^+	r	I^-	r	I^+	r
8	1.08e-01	-	1.07e-02	-	2.23e-03	-	7.53e-06	-
32	5.57e-02	0.96	5.08e-03	1.07	6.02e-04	1.89	8.95e-07	3.07
128	2.11e-02	1.4	2.47e-03	1.04	1.56e-04	1.95	1.09e-07	3.04
	$p=2$				$p=3$			
N	I^-	r	I^+	r	I^-	r	I^+	r
8	7.56e-05	-	1.98e-08	-	3.02e-06	-	8.26e-11	-
32	1.07e-05	2.82	4.83e-10	5.35	2.25e-07	3.74	6.17e-13	7.06
128	1.43e-06	2.93	1.77e-11	4.77	1.55e-08	3.85	5.44e-15	6.82

Table 2: Average errors and estimated convergence rates on the inflow I^- and outflow I^+ boundaries L for Example 1 as a function of p and N.

Example 2. Consider the initial-boundary value problem for the inviscid Burgers' equation

$$u_y + (\frac{u^2}{2})_x = 0, \qquad u(x,0) = \frac{1+\sin \pi x}{2}, \qquad u(-1,y) = u(1,y) \tag{50}$$

on $-1 \leq x < 1$, $0 < y \leq 0.3$. The problem illustrates the performance of the error estimation procedure on a problem with a stronger nonlinearity. The sinusoidal data steepens as y increases and would form a shock if the domain were larger in the y direction. At this point, our error estimates would cease to apply.

Computations were again performed on a series of unstructured meshes with p ranging from 0 to 4. We present effectivity indices in Figure 2. The error estimates are within 15% of actual for all but the coarsest mesh spacings.

4 Discussion

We describe a simple procedure for constructing discretization error estimates for hyperbolic problems solved by discontinuous Galerkin methods. The approach extends earlier work of Adjerid *et al.* [3] from one to two spatial dimensions and to unstructured mesh problems. The error estimates only involve local (elemental) computations. We exhibit a higher rate of convergence (on average) at the downwind end(s) of elements that has enabled us to obtain global as well as local error estimates.

There are several limitations to the current theory which should be overcome. Some of these are relatively simple and were omitted to maintain a clarity of the presentation. They include three- or higher-dimensional problems. Others, such as problems where coefficients change rapidly or nonlinear problems with discontinuities offer significant challenges and may, indeed, not be amenable to our approach.

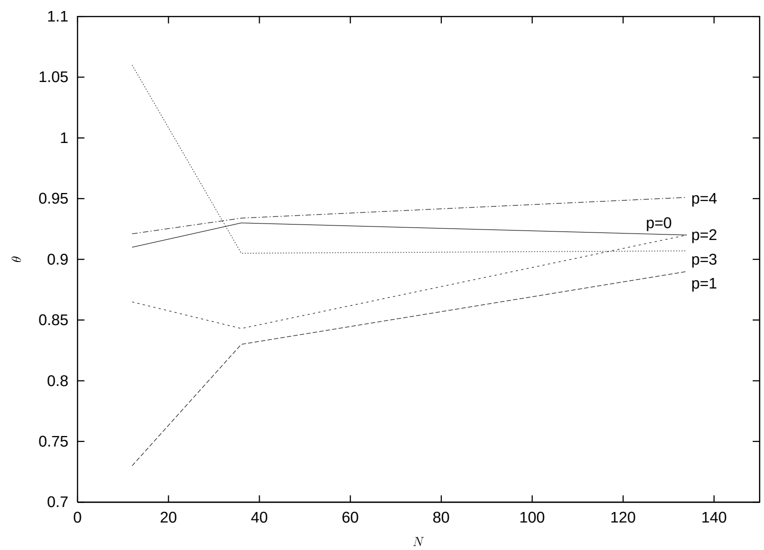

Figure 2: Effectivity indices for the Burgers' equation.

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