

Adaptive Discontinuous Galerkin Method for the Shallow Water Equations

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SUMMARY

In this paper, we present a Discontinuous Galerkin formulation of the shallow water equations. An orthogonal basis is used for the spatial discretization and an explicit Runge-Kutta scheme is used for time discretization. Some results of second order anisotropic adaptive calculations are presented for dam breaking problems. The adaptive procedure uses an error indicator that concentrates the computational effort near discontinuities like hydraulic jumps. Copyright © 2003 John Wiley & Sons, Ltd.

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1. Introduction

The Discontinuous Galerkin Method (DGM) was initially introduced by Reed and Hill in 1973 [19] as a technique to solve neutron transport problems. Recently, the DGM has become popular and it has been used for solving a wide range of problems [5].

The DGM is a finite element method in the sense that it involves a double discretization. First, the physical domain Ω is discretized into a collection of \mathcal{N}_e elements

$$\mathcal{T}_e = \bigcup_{i=1}^{\mathcal{N}_e} \Omega_i \quad (1)$$

called a mesh. Then, the continuous function space $V(\Omega)$ containing the solution u of a given PDE is approximated using a finite expansion into polynomials in space and finite differences in time. The accuracy of a finite element discretization depends both on geometrical and functional discretizations. Adaptivity seeks an optimal combination of these two ingredients:

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p-refinement is the expression used for functional enrichment and h-refinement for mesh enrichment. In this paper, we will only focus on h-refinement.

The shallow water equations (SWEs), which describe the inviscid flow of a thin layer of fluid in two dimensions, have been used for many years by both the atmospheric modeling and by the hydraulic communities as a vehicle for testing promising numerical methods for solving atmospheric, oceanic, dam breaking and river flow problems.

It is only recently that the DGM has been applied to shallow water equations. Schwanenberg et al. [23] have developed a local DGM for shallow water equations where they use the Harten and Lax numerical flux [14]. In [11], Giraldo et al. use a fast quadrature free DGM for solving the spherical SWEs. In this, our aim is to show how h-adaptivity is able to provide highly accurate solutions of the SWEs. For the spatial discretization of the unknowns, we choose an orthogonal basis that diagonalizes the mass matrix and, thus, simplifies its evaluation. The free surface allows gravity wave (sound waves) propagation at speed $c_g = \sqrt{gh}$ where h is the fluid depth and g is the acceleration of gravity. In our examples, water depth is sufficiently small so that $c_g = \mathcal{O}||\mathbf{v}||$ where \mathbf{v} is the fluid velocity. Typically, when the sound waves have the same propagation speed as the material waves, an explicit time integration is well adapted for computing. We use here a second order explicit TVD Runge-Kutta time integration scheme [4].

Transient computation of flows including moving features like abrupt wave fronts are applications where adaptivity in time is crucial. Without explicit interface tracking [10], h-adaptivity will certainly be necessary to accurately represent the complex evolution of waves. We present procedures to perform adaptive computations where the discretization space V_h changes in time. Both conforming and non-conforming adaptation schemes are presented and compared with known results and experiments.

2. Discontinuous Galerkin Formulation of Shallow Water Equations

2.1. Continuous Formulation

Consider an open set $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ is Lipschitz continuous with a normal \vec{n} that is defined everywhere. We seek to determine $\mathbf{u}(\Omega, t) : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbf{L}^2(\Omega)^m = V(\Omega)$ as the solution of a *system of conservation laws*

$$\partial_t \mathbf{u} + \mathbf{div} \vec{\mathbf{F}}(\mathbf{u}) = \mathbf{r}. \quad (2)$$

Here $\mathbf{div} = (\nabla \cdot, \dots, \nabla \cdot)$ is the vector valued divergence operator and

$$\vec{\mathbf{F}}(\mathbf{u}) = (\vec{F}_1(\mathbf{u}), \dots, \vec{F}_m(\mathbf{u}))$$

is the flux vector with the i th component $\vec{F}_i(\mathbf{u}) : (\mathbf{H}^1(\Omega))^m \rightarrow \mathbf{H}(\text{div}, \Omega)$. Function space $\mathbf{H}(\text{div}, \Omega)$ consists of square integrable vector valued functions whose divergence is also square integrable i.e.,

$$\mathbf{H}(\text{div}, \Omega) = \left\{ \vec{v} \mid \vec{v} \in \mathbf{L}^2(\Omega)^2, \nabla \cdot \vec{v} \in \mathbf{L}^2(\Omega) \right\}.$$

With the aim of constructing a Galerkin form of (2), multiply equation (2) by a test function $\mathbf{w} \in V(\Omega)$, integrate over Ω to obtain

$$\int_{\Omega} \partial_t \mathbf{u} \cdot \mathbf{w} \, dv + \int_{\Omega} \mathbf{div} \vec{\mathbf{F}}(\mathbf{u}) \cdot \mathbf{w} \, dv = \int_{\Omega} \mathbf{r} \cdot \mathbf{w} \, dv, \quad \forall \mathbf{w} \in V(\Omega). \quad (3)$$

We then use the divergence theorem to obtain the following variational formulation

$$\int_{\Omega} \partial_t \mathbf{u} \cdot \mathbf{w} \, dv - \int_{\Omega} \vec{\mathbf{F}}(\mathbf{u}) \cdot \nabla \mathbf{w} \, dv + \int_{\partial\Omega} \vec{\mathbf{F}}(\mathbf{u}) \cdot \vec{\mathbf{n}} \mathbf{w} \, ds = \int_{\Omega} \mathbf{r} \cdot \mathbf{w} \, dv, \quad \forall \mathbf{w} \in V(\Omega). \quad (4)$$

2.2. Discrete Formulation

Finite element methods (FEMs) involve a double discretization. First, the physical domain Ω is discretized into a collection of \mathcal{N}_e elements

$$\mathcal{T}_e = \bigcup_{e=1}^{\mathcal{N}_e} e \quad (5)$$

called a mesh. This first step is the one of *geometrical discretization*.

Then, the continuous function spaces (infinite dimensional) are replaced by finite dimensional expansions. The difference between the DGM and classical FEMs is that the solution is approximated in each element separately: No *a priori* continuity requirements are needed. The discrete solution may then be discontinuous at inter-element boundaries. Figure 1 shows a typical situation of three elements e_1 , e_2 and e_3 . The approximated field u is smooth in each element but may be discontinuous at inter-element boundaries.

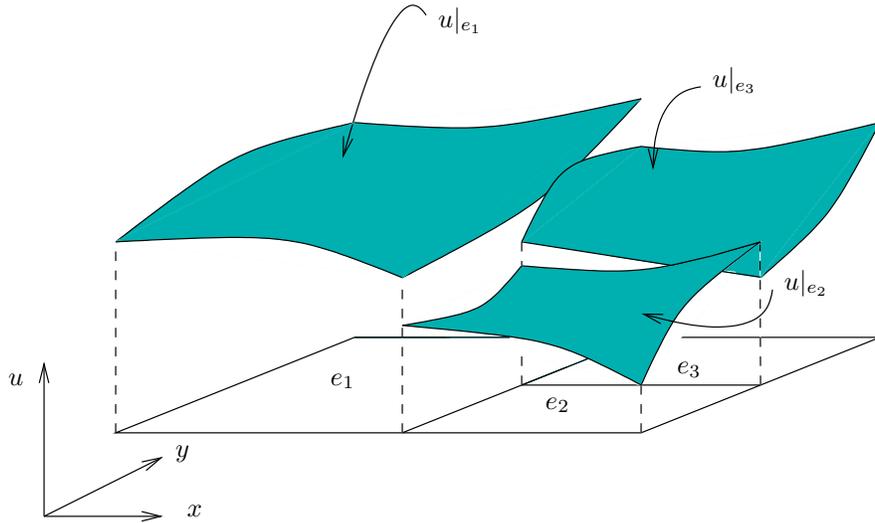


Figure 1. Three elements e_1 , e_2 and e_3 and the piecewise discontinuous solution u .

In each element, we have m field components u_i , $i = 1, \dots, m$. In the case of shallow water equations, we have $m = 3$ unknown fields: The water height h and the 2 components of the fluid velocity v_x and v_y . Here, each field is approximated with the same discrete space. This simplification is done without any loss of generality.

In each element, it is usually polynomial spaces that are chosen for approximating the u_i 's. We note

$$\mathbb{P}^k = \text{span} \{x^l y^m, 0 \leq l, ml + m \leq k\}$$

the space of complete polynomials of total degree at most k . The dimension $d = \dim \mathbb{P}^k$ is $d = (k+1)(k+2)/2$. Again, we note

$$\mathbb{Q}^k = \text{span} \{x^l y^m, 0 \leq l, m \leq k\}$$

the space of complete polynomials of degree at most k . The dimension $d = \dim \mathbb{Q}^k$ is $d = (k+1)^2$. The approximation of component u_i over element e , noted u_i^e is written

$$u_i^e \simeq \sum_{j=1}^d \phi_j(x, y) U(e, i, j)$$

where the $U(e, i, j)$ are the coefficients of the approximation or degrees of freedom. In each element, we have $m \times d$ coefficients, and, because we consider that all approximations are disconnected, there are $\mathcal{N}_e \times m \times d$ coefficients for the whole mesh. We also have

$$B = \{\phi_1, \dots, \phi_d\}$$

that is a basis of \mathbb{P}^k if e is a triangle, or, of \mathbb{Q}^k if e is a quadrangle. Usual FEM have limited choices for B due to the continuity requirements of the approximation. Nodal, hierarchical, Serendip or non-conforming basis are among the usual basis for classical FEMs. In case of the DGM, there are no limitations for the choice of the ϕ_i 's. In fact, any basis B can be used and, in terms of the numerical solution, all basis for the same polynomial degree k are strictly equivalent i.e. lead to the same numerical solution. For example, \mathbb{P}^1 can be spanned with $B = \{1, x, y\}$, $B = \{1 - x - y, x, y\}$ or any other combination. Even if the numerical solution does not depend on the choice of the basis B , it is advantageous to use bases [20] that have the following L^2 -orthogonality property

$$\int_e \phi_i \phi_j \, dv = \delta_{ij}.$$

For quadrangles, orthogonal bases are constructed as tensor products of Legendre polynomials. For triangles, Dubiner [7] or Remacle [20] have developed orthogonal basis through Gram-Schmidt orthogonalization. For example, one of the many L^2 -orthogonal expansion of \mathbb{P}^1 can be written in the canonical triangle as

$$B = \left\{ \sqrt{2}, -2 + 6x, -2\sqrt{3}(1 - x - 2y) \right\}.$$

For each field component u_i^e in element e , we have to solve the following d equations (one equation per ϕ_j)

$$\int_e \partial_t u_i^e \phi_j \, dv - \int_e \left[\vec{F}_i(\mathbf{u}^e) \nabla \phi_j + r_i \phi_j \right] \, dv + \int_{\partial e} \vec{F}_i(\mathbf{u}) \cdot \vec{n} \phi_j \, ds = 0 \quad \forall \phi_j. \quad (6)$$

If the ϕ_j 's are orthonormal over the reference element and if V_e is the volume of element e , (6) simplifies into

$$V_e \partial_t U(e, i, j) = \int_e \left[\vec{F}_i(\mathbf{u}^e) \nabla \phi_j + r_i \phi_j \right] \, dv - \int_{\partial e} \vec{F}_i(\mathbf{u}) \cdot \vec{n} \phi_j \, ds \quad \forall \phi_j. \quad (7)$$

Note that equation (7) is correct only if elements have constant jacobians (straight sided triangles e.g.). Now, a discontinuous basis implies that \mathbf{u} is not unique on ∂e and, consequently,

that the normal trace $\vec{F}_i \cdot \vec{n}$ is not defined on ∂e . In this situation, a *numerical flux* f_i is usually used on each portion ∂e_k of ∂e shared by element e and neighboring element e_k . Here, \mathbf{u}^e and \mathbf{u}^{e_k} are the restrictions of solution \mathbf{u} , respectively, to element e and element e_k . With a numerical flux, equations (7) becomes

$$V_e \partial_t U(e, i, j) = \int_e \left[\vec{F}_i(\mathbf{u}^e) \nabla \phi_j + r_i \phi_j \right] dv - \sum_{k=1}^{n_e} \int_{\partial e_k} f_i(\mathbf{u}^e, \mathbf{u}^{e_k}) \phi_j ds \quad \forall \phi_j. \quad (8)$$

where n_e is the number of faces of element e .

2.3. Shallow water equations

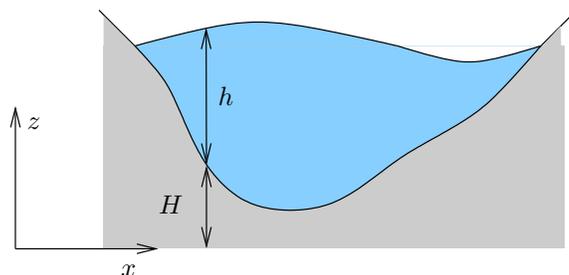


Figure 2. The water bed depth H and the water height h .

The movement of an incompressible fluid with constant density under the influence of a gravitational body force is considered. The description is basically inviscid except for the possible inclusion of a viscous bottom friction term. Vertical accelerations of the fluid are neglected, which allows integrating the remaining part of the vertical momentum equation and to obtain an expression for the pressure which in turn can then be eliminated from the system. The error associated with this approximation is of the order of h^2/l^2 (h undisturbed water height, l characteristic length scale of the waves in x -direction). This estimate is equivalent to the so-called “long-wave limit” of wave motion, i.e. we are dealing with either very long waves or with shallow water. Physically, the horizontal velocity that is retained can be interpreted as a vertical average of the fluid velocity. The shallow water equations have the form (2) with $m = 3$,

$$\mathbf{u} = \begin{Bmatrix} h \\ hv_x \\ hv_y \end{Bmatrix}, \quad \vec{\mathbf{F}}(\mathbf{u}) = \begin{Bmatrix} h\vec{v} \\ hv_x\vec{v} + \frac{1}{2}gh^2\vec{e}_x \\ hv_y\vec{v} + \frac{1}{2}gh^2\vec{e}_y \end{Bmatrix}$$

and

$$\mathbf{r} = \begin{Bmatrix} 0 \\ -gh(\partial_x H + S_{fx}) \\ -gh(\partial_y H + S_{fy}) \end{Bmatrix}.$$

Here h is the water depth above the bed, H is the bed elevation (see Figure 2), \vec{v} the water velocity, g the acceleration of gravity, \vec{e}_x and \vec{e}_y are the unit vectors in the x and y directions,

respectively and $v_x = \vec{v} \cdot \vec{e}_x$ and $v_y = \vec{v} \cdot \vec{e}_y$. The two components of the bottom friction S_{fx} and S_{fy} can be expressed by the Manning formula

$$S_{fx} = \frac{n^2 v_x \sqrt{v_x^2 + v_y^2}}{h^{4/3}}, \quad S_{fy} = \frac{n^2 v_y \sqrt{v_x^2 + v_y^2}}{h^{4/3}}. \quad (9)$$

where n is an empirical roughness coefficient.

2.4. Numerical Flux

We concentrate on the time evolution of our flow model from an initial state that consists of two semi-infinite uniform zones which are separated by a discontinuity. This setup is usually referred as a *Riemann problem* [25, 16]. One can imagine a realization of this situation by positioning a diaphragm (“an infinitely thin dam”) between the two fluid states and somehow rupturing it at time $t = 0$. Our objective is to determine the resulting induced wave motion as a function of the initial state. This problem is geometrically one-dimensional in that the solution only depends on one space coordinate normal to the diaphragm. The Riemann problem related to non linear hyperbolic systems is usually hard to compute because of the non linearity of fluxes. Usually, it is only available numerically through a Newton-Raphson iteration.

The idea of using the solution of Riemann problems for solving non linear conservation laws numerically is from Godunov [13]. At the time, it was of course in the context of finite differences. Following the idea of Godunov, we consider that each edge ∂e of the mesh is a diaphragm separating two states: The solutions \mathbf{u} in the two elements neighboring ∂e . The numerical flux \mathbf{f} is then computed using the solution \mathbf{u}^R of the associated Riemann problem:

$$\mathbf{f} = \vec{\mathbf{F}}(\mathbf{u}^R) \cdot \vec{n}.$$

For linear problems, this technique consists in taking the “fully upwind flux” and therefore the resulting scheme is stable. For non linear problems, it has been proven that the choice of Godunov fluxes are ensuring that the numerical solution satisfies the entropy condition [16].

Practically, solving exactly Riemann problems at each mesh edge is complex and computationally prohibitive. In most of the cases (except perhaps in the context of high speed compressible flows with very strong shocks [28]), an approximate solution to the Riemann problem is sufficient. Approximate Riemann solvers produce more numerical dissipation than Godunov fluxes. Hence, numerical experience suggests that the choice of a given numerical flux (that respects a discrete entropy condition) does not have a significant impact on the accuracy of the solution, especially when polynomial degree k increases.

Among the approximate Riemann solvers designed for computing the numerical flux \mathbf{f} , a very pragmatic and successful approach has been taken by Roe [22] and later extended to the shallow water equations [12, 2]. The approach consists in constructing the exact solution of a linearized Riemann problem at each mesh edge. In the framework of shallow-water equations applied to water-wave propagation problems, an important parameter is the Froude number Fr , defined as the ratio between the water velocity v and the celerity $c = \sqrt{gh}$. The Froude number determines the subcritical ($\text{Fr} < 1$) or supercritical ($\text{Fr} > 1$) character of the flow. Following Soares Frazão and Zech [9], the Roe numerical flux for shallow-water equations can be written as a function of Fr as

$$\mathbf{f}(\mathbf{u}_e, \mathbf{u}_{e_k}) = \frac{1}{2} \left[(1 + \text{Fr}_A) \vec{\mathbf{F}}(\mathbf{u}_e) + (1 - \text{Fr}_A) \vec{\mathbf{F}}(\mathbf{u}_{e_k}) \right] \cdot \vec{n} + \frac{1}{2} c_A (1 - \text{Fr}_A^2) [\mathbf{u}_e - \mathbf{u}_{e_k}] \quad (10)$$

where the subscript A denotes average quantities. In (10), the average Froude number

$$\text{Fr}_A = \frac{\vec{v}_A \cdot \vec{n}}{c_A}$$

is computed using Roe averages i.e.

$$(v_x)_A = \frac{(v_x)_e \sqrt{h_e} + (v_x)_{e_k} \sqrt{h_{e_k}}}{\sqrt{h_e} + \sqrt{h_{e_k}}}, \quad (v_y)_A = \frac{(v_y)_e \sqrt{h_e} + (v_y)_{e_k} \sqrt{h_{e_k}}}{\sqrt{h_e} + \sqrt{h_{e_k}}},$$

$$\vec{v}_A = \{(v_x)_A, (v_y)_A\}^T \quad \text{and} \quad c_A = \sqrt{g \frac{1}{2} (h_e + h_{e_k})}.$$

In order to use (10) for both the sub- and supercritical case, the absolute value of the Froude number Fr is bounded by 1. We correct its value in order to fulfill this physical constraint: If $\text{Fr} > 1$, $\text{Fr} = 1$ and if $\text{Fr} < -1$, $\text{Fr} = -1$. The situation is similar in the case of the Euler equations, as noted in the original paper by Roe [22]. The flux is the sum of a high order centered term plus a dissipation term of order zero:

$$\mathbf{f} = \underbrace{\frac{1}{2} [\vec{\mathbf{F}}(\mathbf{u}_e) + \vec{\mathbf{F}}(\mathbf{u}_{e_k})]}_{\text{Centered differences}} \cdot \vec{n} + \underbrace{\frac{1}{2} \text{Fr} [\vec{\mathbf{F}}(\mathbf{u}_e) - \vec{\mathbf{F}}(\mathbf{u}_{e_k})]}_{\text{Dissipation}} \cdot \vec{n} + \frac{1}{2} c_A (1 - \text{Fr}^2) [\mathbf{u}_e - \mathbf{u}_{e_k}].$$

It has been proven in [1] that, in smooth regions, $\mathbf{u}_e - \mathbf{u}_{e_k} = \mathcal{O}(h^{k+1})$ so that the dissipation introduced by the upwinding does not exceed the truncature error of the scheme. Near discontinuities or in badly resolved zones where the mesh is too coarse, $\mathbf{u}_e - \mathbf{u}_{e_k} = \mathcal{O}(h)$ and the scheme is first order, as expected.

3. Anisotropic Mesh Adaptation

3.1. Definition of the Metric Field \mathcal{M}

The goal of our mesh adaptation process is to determine the anisotropic mesh configuration that will most effectively provide the level of accuracy required for the parameters of interest. The strategy adopted in the present paper is to construct an optimal anisotropic mesh through a metric field $\mathcal{M}(\vec{x})$, $\vec{x} \in \Omega$ defined over the domain of the analysis Ω . The goal of a mesh adaptation is to build a mesh where every edge is of size 1, using the non uniform measure of distance defined by the metric \mathcal{M} . If \vec{y} is the vector representing an edge with its Euclidian coordinates y_1, y_2, y_3 , the optimum mesh is characterized by:

$$\vec{y}^T \mathcal{M} \vec{y} = 1 \quad \forall \vec{y}.$$

The metric \mathcal{M} is a symmetric covariant tensor with all its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ positive. If $\vec{E}_1, \vec{E}_2, \vec{E}_3$ are the orthogonal unit eigenvectors associated with the eigenvalues $\lambda_1, \lambda_2, \lambda_3$, the metric tensor can be written as

$$\mathcal{M} = \mathcal{R}^T \Lambda \mathcal{R}$$

with

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$$

and

$$\mathcal{R} = \begin{bmatrix} \vec{E}_1 \\ \vec{E}_2 \\ \vec{E}_3 \end{bmatrix}.$$

The interpretation of mesh optimality in terms of unit edges is then equivalent to

$$\vec{y}^T \mathcal{R}^T \Lambda \mathcal{R} \vec{y} = 1 \quad \forall \vec{y}.$$

where $\mathcal{R}\vec{y} = \{Y_1, Y_2, Y_3\}^T$ are the coordinates of \vec{y} in the principal axis of the metric \mathcal{M} i.e. (see Figure 3)

$$\vec{y} = Y_1 \vec{E}_1 + Y_2 \vec{E}_2 + Y_3 \vec{E}_3.$$

The optimality is then re-written as

$$Y_1^2 \lambda_1 + Y_2^2 \lambda_2 + Y_3^2 \lambda_3 = 1$$

which means that, at each point of the domain, an edge is of optimal size if its extremity sits on an ellipsoid of equation $Y_1^2 \lambda_1 + Y_2^2 \lambda_2 + Y_3^2 \lambda_3 = 1$ (see Figure 3). The eigenvalues of the

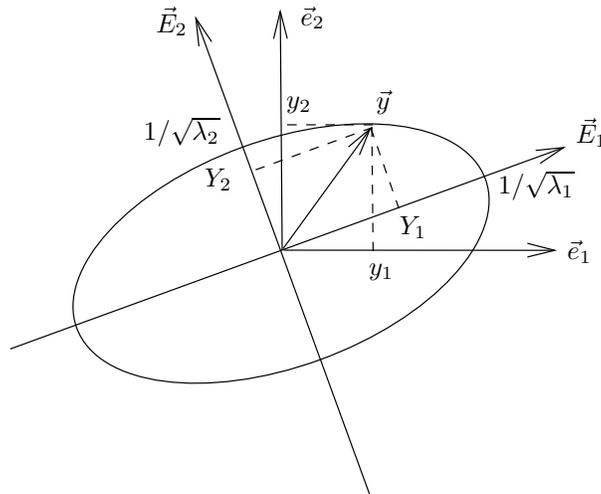


Figure 3. Interpretation in 2D of the different parameters in the definition of the metric \mathcal{M} .

metric are then directly related to desired mesh sizes in the principal directions of the metric \mathcal{M} .

3.2. Computing mesh sizes

Those 3 sizes $s_1 = 1/\sqrt{\lambda_1}$, $s_2 = 1/\sqrt{\lambda_2}$ and $s_3 = 1/\sqrt{\lambda_3}$ are computed using an error indication procedure. In smooth regions, i.e. regions far from hydraulic jumps, the error is computed using the tensor of second order derivatives of the water height h :

$$H_{ij} = \frac{\partial^2 h}{\partial x_i \partial x_j}.$$

The Hessian H is computed at each vertex using a patchwise linear reconstruction of the gradients ∇h [21]. The Hessian is symmetric and we have then the orthogonal decomposition

$$H = \mathcal{R}^T \bar{\Lambda} \mathcal{R}$$

with

$$\bar{\Lambda} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)$$

In terms of Taylor expansion theorem, the discretization error e in the direction of the i th eigenvector of H is proportional to $\bar{s}_i^2 \bar{\lambda}_i$, i.e. $e = c \bar{s}_i^2 \bar{\lambda}_i$, where \bar{s}_i is the local mesh edge length in the i th direction for previous computation. We can compute a mesh with equidistributed error by choosing

$$s_i^2 = \frac{\epsilon}{c \bar{s}_i^2 |\bar{\lambda}_i|}$$

where ϵ is a targetted error that is defined by the user, c is the coefficient of proportionality and s_i is desired local mesh edge length in the i th direction. It is usual to defined maximal and minimal element sizes s_{max} and s_{min} that prevent to build up non realistic metric fields. The metric is then corrected by

$$\lambda_i = \max \left(\min \left(\frac{1}{s_i^2}, \frac{1}{s_{min}^2} \right), \frac{1}{s_{max}^2} \right).$$

In [15], we have shown how to detect efficiently non smooth regions in DGM computations. Our discontinuity detector has the following form:

$$\mathcal{I}_e = \frac{\sum_{j=1}^{n_e} \left| \int_{\partial e_j} (h^e - h^{e_j}) ds \right|}{s_e^{(k+1)/2} |\partial e| \|h^e\|}. \quad (11)$$

In all forthcoming examples, we choose s_e as the radius of the circumscribed circle in element e , and use a maximum norm based on local solution maxima at integration points.

We can show that $\mathcal{I}_e \rightarrow 0$ as either $s_e \rightarrow 0$ or $p \rightarrow \infty$ in smooth solution regions, whereas $\mathcal{I}_e \rightarrow \infty$ near a discontinuity. Thus, the discontinuity detection scheme is

$$\begin{cases} \text{if } \mathcal{I}_e > 1, & h \text{ is discontinuous} \\ \text{if } \mathcal{I}_e < 1, & h \text{ is smooth} \end{cases}. \quad (12)$$

Near discontinuities, Hessians are highly ill conditioned. When a discontinuity is detected, we use the gradients ∇h in order to compute the metric field. Because we know that, without diffusion, a hydraulic jump has an infinitely small thickness, we use the minimal allowable mesh size s_{min} in the direction of the gradient and the maximal size s_{max} on the other direction.

A metric smoother is crucial at this point in order to reconnect smooth and discontinuous regions. The algorithm is described in [21].

3.3. Mesh Adaptation using Local Mesh modifications

In our approach, the mesh is adapted using local mesh modification operators that enable fast and accurate solution transfers. Given the mesh metric field \mathcal{M} defined over the domain, the goal is to apply local mesh modification operators to yield a mesh of the same quality as would be obtained by an anisotropic domain remeshing procedure. In every mesh modification

operator, one cavity triangulation \mathcal{C} , i.e. a set of mesh entities that form a connected volume, is replaced by another cavity triangulation \mathcal{C}' with the same closure. Formally, we write

$$\mathcal{T}^{n+1} = \mathcal{T}^n + \mathcal{C}' - \mathcal{C}. \quad (13)$$

where \mathcal{T}^n denotes the mesh before the local mesh modification and \mathcal{T}^{n+1} is the mesh after the local mesh modification. In the kernel of each mesh modification, we ensure that there is a moment when both cavity triangulations are present. We are able then to rewrite (13) as a two step procedure:

$$\mathcal{T}' = \mathcal{T}^n + \mathcal{C}'. \quad (14)$$

$$\mathcal{T}^{n+1} = \mathcal{T}' - \mathcal{C}. \quad (15)$$

\mathcal{T}' represents a mesh that is topologically incorrect but both cavity triangulations \mathcal{C} and \mathcal{C}' are present so that we can insert a callback to the solver to transfer the solution \mathbf{u} from \mathcal{C} to \mathcal{C}' . One mesh modification plus solution transfer operation is done in 3 steps:

$$\mathcal{T}' = \mathcal{T}^n + \mathcal{C}'. \quad (16)$$

$$\Pi_{L^2}(\mathbf{u}, \mathcal{C}, \mathcal{C}') \quad (17)$$

$$\mathcal{T}^{n+1} = \mathcal{T}' - \mathcal{C}. \quad (18)$$

where $\Pi_{L^2}(\mathbf{u}, \mathcal{C}, \mathcal{C}')$ is, in our case, the L^2 projection of the solution \mathbf{u} from the cavity \mathcal{C} to \mathcal{C}' . The L^2 projection of the the solution from \mathcal{C} to \mathcal{C}' ensures that conservative quantities like mass or momentum's are conserved through the solution transfer process.

In two dimensions, we only use the three mesh modification operators that are described below.

3.3.1. Edge splitting The first mesh modification operator is the edge splitting. It consists in modifying a cavity \mathcal{C} composed of 2 neighboring elements e_1 and e_2 by splitting the common edge and replacing e_1 and e_2 by e_3, e_4, e_5 and e_6 as represented in Figure4.

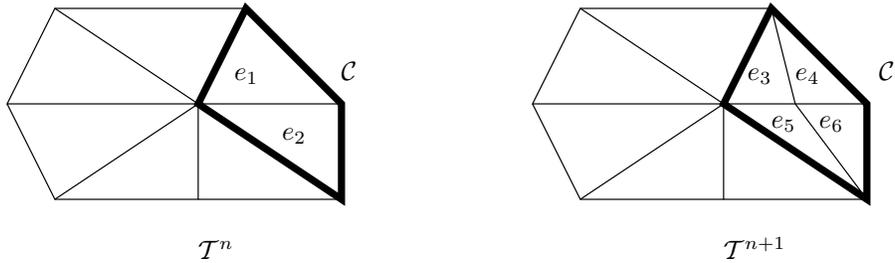


Figure 4. Edge splitting in 2D.

3.3.2. Edge collapsing The second mesh modification operator is the edge collapsing. An edge \vec{y} is too short so it is decided to remove it. The origin of the edge is moved to its end, modifying a cavity \mathcal{C} composed of all the triangles neighboring the origin of the edge (see Figure5).

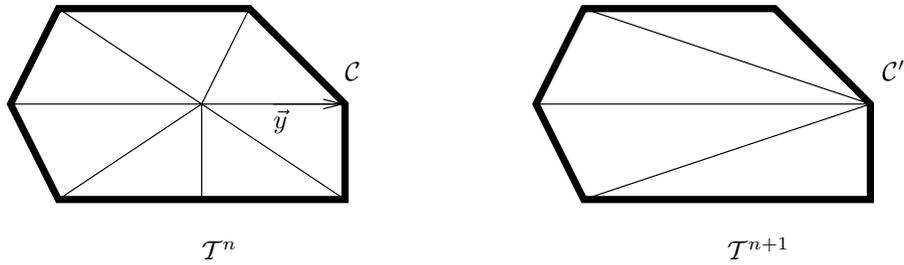


Figure 5. Edge collapsing in 2D.

3.3.3. Edge swapping The last mesh modification operator is the edge swapping. It consists in modifying a cavity \mathcal{C} composed of 2 neighboring elements e_1 and e_2 by swapping the edge and replacing e_1 and e_2 by e_3 and e_4 as represented in Figure 6.

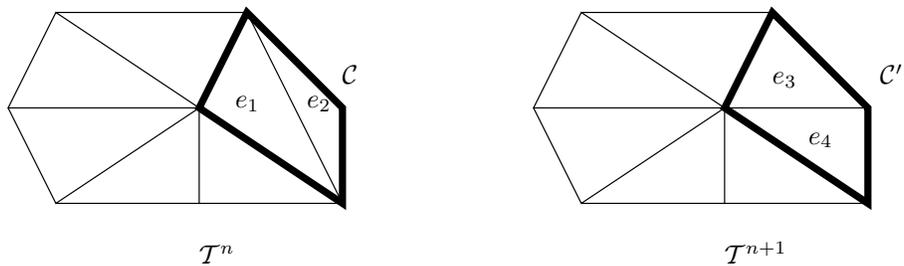


Figure 6. Edge swapping in 2D. An edge separating 2 triangles is replaced by the opposite edge.

The mesh adaptation algorithm for the 3-dimensional case is described in [18]. Edge collapsing's are used to coarsen the mesh in regions of low error while edge splitting's are used to refine the mesh in regions of high error. Edge swapping's are used to align the mesh to the metric axes and to enhance mesh quality. Note that we do not consider here vertex repositioning in our mesh modification procedure. The consequence of the numerous local mesh modifications on the quality of the numerical solution is of importance. Questions are

- Does the adaptive procedure introduce excessive numerical dissipation?
- Does the adaptive procedure introduce oscillations?
- Does the adaptive procedure introduce some loss of conservativity?

The edge splitting procedure does not change the numerical solution. Consequently it does not introduce any complexity.

While performing edge collapsing, some information is lost because the discrete space where the numerical solution lives is made smaller. Some numerical dissipation may then be introduced by the operation. The L^2 projection $\Pi_{L^2}(\mathbf{u}, \mathcal{C}, \mathcal{C}')$ may also introduce Gibbs oscillations (overshoots) for polynomial order $k > 0$. Because we use L^2 projections, the conservative quantities are transferred correctly from \mathcal{C} to \mathcal{C}' . Note that, for the target cavity \mathcal{C}' , the solution over \mathcal{C} is piecewise discontinuous and, consequently, we have to be careful and compute the L^2 projection accurately. Despite of that, we do not expect that edge collapsing's

will cause any problem since they are performed in regions of low error only and, therefore, the resulting modifications of the solution due to the operation should not have any significant impact.

The edge swapping is certainly the most controversial. Edge swappings are performed everywhere, including regions of high error. Similarly to edge collapsing, edge swapping is an operation that modifies the solution and therefore introduces errors. Conservation is not an issue because we do not consider curved geometries here. This issue will be considered in a forthcoming paper.

4. Results

4.1. Radial dam break problem.

Consider the SWEs with piecewise initial data

$$h(r) = \begin{cases} 2 & \text{if } r < 1 \\ 1 & \text{if } r > 1 \end{cases},$$

$r = \sqrt{x^2 + y^2}$ being the radial coordinate. The fluid is initially at rest i.e. $v_x = v_y = 0$. This is the problem of a dam break i.e. two zones at rest (null velocity) that are separated by an interface (the dam), the water height being different in these two zones. At $t = 0$, the interface is impulsively removed (the dam breaks). The dam break Riemann problem for SWEs has a solution that consists in a rarefaction wave and a hydraulic jump (shock). The rarefaction wave moves in the direction of the highest water depth (i.e. radially, towards the origin) while the shock moves in the direction of low water depth. Once the rarefaction wave hits the origin $r = 0$, it is reflected and the fluid flows back outwards. A second shock rapidly forms. This problem has an interesting structure and will be used to verify our refinement methodology:

- Examine the influence of multiple mesh refinement and coarsening steps on the preservation of conservation (i.e. verify if the adaptive scheme is able to compute the right speed for the waves);
- Show that the anisotropic refinement produces optimal discretizations (i.e. verify that anisotropic refinement is cheaper than isotropic refinement or no refinement for a given level of solution resolution).

For that purpose, we need the exact solution of the problem. There is no analytical solution for the radial dam break problem. Our reference solution was computed numerically by solving the one dimensional equations

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial}{\partial r}(hv_r) &= -\frac{hv_r}{r} \\ \frac{\partial v_r}{\partial t} + \frac{\partial}{\partial r}\left(hv_r^2 + \frac{1}{2}gh^2\right) &= -\frac{hv_r^2}{r} \end{aligned} \quad (19)$$

with a high resolution finite volume scheme on a very fine grid of 2000 points [17]. In what follows, we will call it the “reference” solution.

We have done three computations of the problem. The first one has been done using a uniform mesh with element sizes of $1/200^{\text{th}}$ of the size of the domain i.e. 0.025. The uniform

mesh has about 20,000 nodes and 40,000 elements. The second computation was done using isotropic refinement i.e. using the same mesh size in each direction as a function of position and time. The mesh was adapted every 0.01 second. Final time of the computation was $t = 1.25$ so that 125 mesh adaptations were performed. The minimum mesh size $s_{min} = 5/200$ i.e. $1/200^{\text{th}}$ of the size of the problem. The maximum mesh size $s_{max} = 5/20$ i.e. $1/20^{\text{th}}$ of the size of the problem. The third computation was done using anisotropic refinement. Adaptation parameters limits were the same as for the isotropic refinement. The three problems have the same effective discretisation with respect to the smallest mesh size at any location in any direction.

Figures 7 and 10 show meshes and plots of water height h at different time steps for an anisotropic simulation.

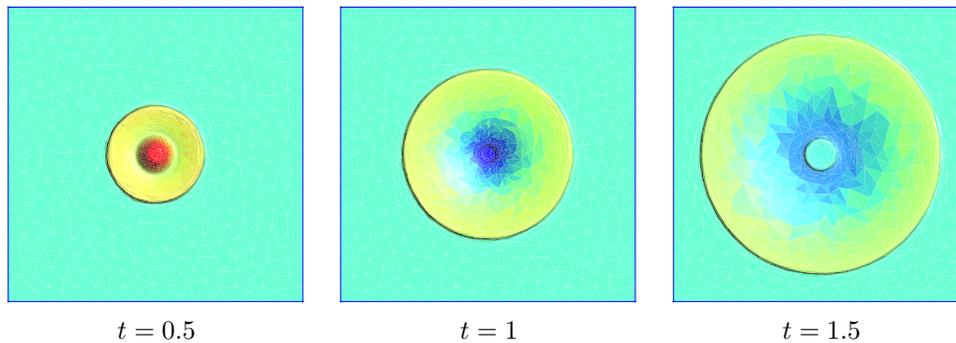


Figure 7. Plots of the water height at different time steps. Those are the solution of the anisotropic run.

Figure 8 shows plots of the water height along radial direction. Those results show that the positions of the waves are not deteriorated by the multiple mesh adaptations. We have shown in §3.3 that the kind of projections used do not cause problems in term of conservation. Another important result is that the solution does not seem to be deteriorated by excessive numerical diffusion: Solutions with and without refinement look similar (this is of course a weak justification that will be refined).

To refine the comparison between solutions with mesh refinement and with the uniform mesh, we have computed (Table 4.1) the L^1 relative norm of the error

$$\epsilon = \frac{\int_0^5 |h - h_{ex}| dr}{\int_0^5 |h_{ex}| dr}$$

for both uniform and adapted meshes (h_{ex} is the “reference” solution computed with the fine 1D grid). We have plotted h and h_{ex} along the radial direction (Figure 8). We see that both uniform and adapted mesh have relative errors ϵ that are comparable.

Of course, there is about 40 times less elements in the anisotropically (see Figure 10) adapted grid than in the uniform grid and about 4 times (see Figure 9) less elements than in the isotropically adapted grid. One can remark that the gain in terms of element number grows with time. At $t = 0.5$, the isotropic mesh has 5386 triangles while the anisotropic one has

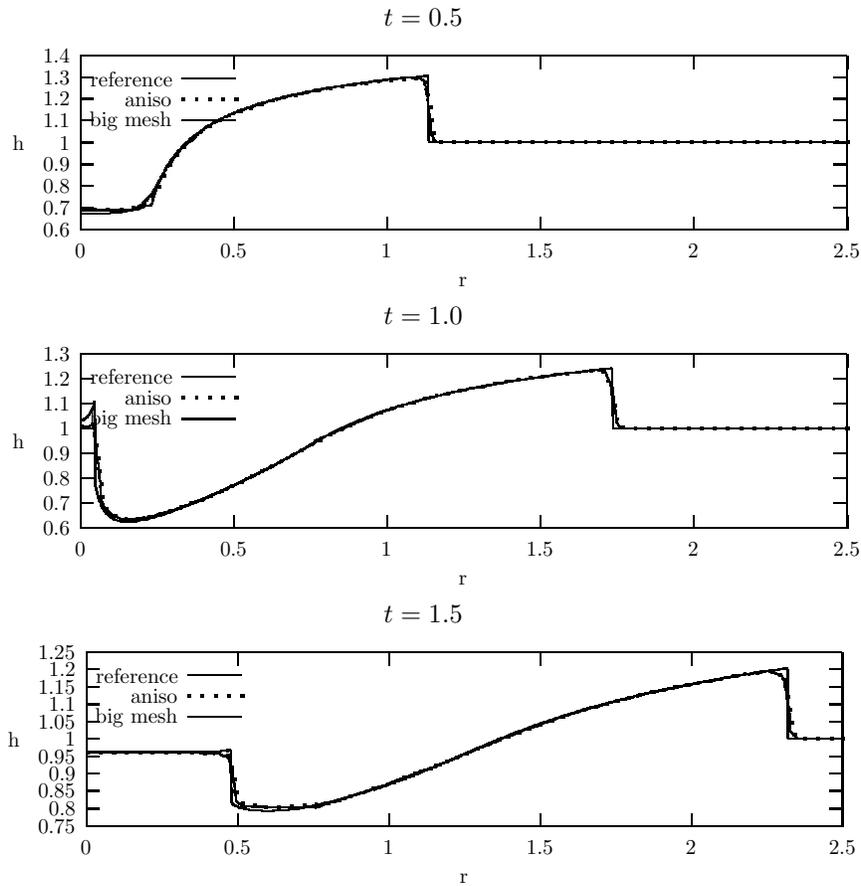


Figure 8. Comparison at different times between exact solution, solution on the uniform grid and solution on the anisotropically adapted grid.

	Uniform mesh	Adapted mesh	Adapted mesh
	Uniform mesh	Isotropic	Anisotropic
$t = 0.00$	1.10758E-03	1.52250E-03	1.56324E-03
$t = 0.25$	3.84628E-03	3.26702E-03	4.12431E-03
$t = 0.50$	3.65177E-03	3.00082E-03	3.42923E-03
$t = 0.75$	4.09531E-03	3.40864E-03	4.07662E-03
$t = 1.00$	4.94453E-03	4.73405E-03	5.09021E-03
$t = 1.25$	3.94322E-03	3.65974E-03	3.85844E-03
$t = 1.50$	3.26890E-03	3.73175E-03	3.66172E-03

Table I. Relative error ϵ for the radial dam break.

2078. At time $t = 1.5$, the isotropic mesh has now 9510 triangles while the anisotropic one only 1932. At early stages of the computation, the radius of curvature of the shock is small so that elements with large aspect ratio are not constructed, at least with the given mesh size parameter s_{min} . Small sizes are required in the radial direction in order to capture the shock and in the azimuthal direction in order to capture the curved shape of the shock. The use of curved elements could certainly help here. When the shock evolves, it grows radially and its curvature decreases. More anisotropic elements can then be constructed and the efficiency of the procedure grows. In fact, it looks that the number of elements in the shock remains almost constant in the process so that the total number of element does not grow in the anisotropic refinement case. In the isotropic case, the number of elements in the shock grows linearly with the distance of the shock to the origin. This effect will be more dramatic in 3D simulations, e.g. for Euler equations of gas dynamics. There, the number of elements in a shock will grow quadratically in the isotropic case while remaining constant in the anisotropic case.

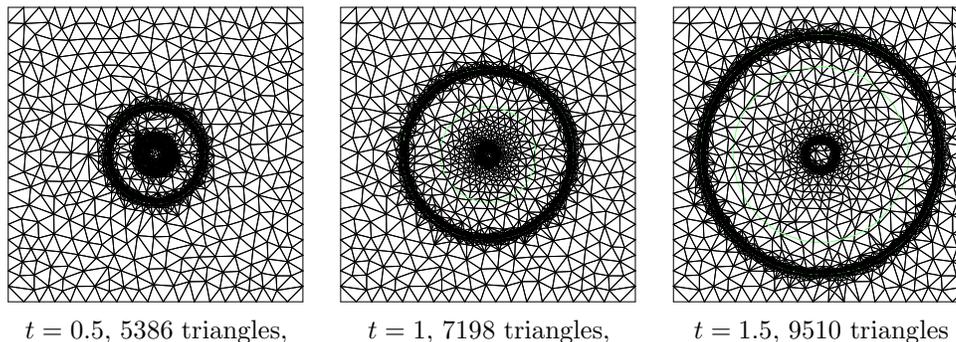


Figure 9. Adapted meshes for the isotropic refinement case.

4.2. Dam breaking in presence of a building

The second example is the one of a dam break wave against a rectangular building [8]. The experiments were performed at the Civil Engineering Department of the Université catholique de Louvain (Belgium), on the setup sketched in Figure 11. The channel has a total length of 35.80 m and is 3.60 m wide. The cross section near the bed is trapezoidal. The building is located 3.40 m downstream from the dam. The initial conditions consist in a 0.40 m water level in the upstream reservoir and a 0.02 m thin water layer in the downstream channel. The friction coefficient n of (9) was chosen as $n = 0.01$.

Two types of flow measurements were performed: Water level and velocity at 5 gauging points, and field data using a Voronoï imaging technique [3, 24]. This latter technique, initially developed for dense granular flow measurements, was used to obtain the experimental pictures that are compared to the computations in some of the following figures. A complete description of the test case and of the available experimental data can be found in [8].

The only computation that we have done for this problem was anisotropically adapted. Parameters of the simulation were $s_{min} = 0.01\text{m}$ and $\epsilon = 0.05$. For sake of comparison, we have generated a uniform triangular mesh with a uniform mesh size of 0.01m. This “equivalent”

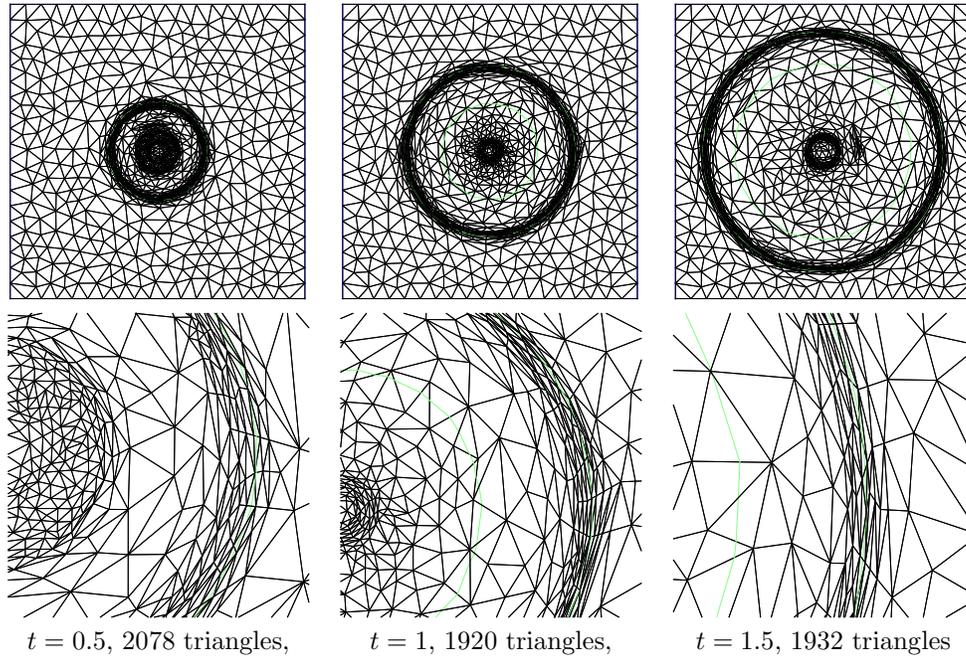


Figure 10. Adapted meshes for the anisotropic refinement case. The images on bottom show zooms of the mesh near the hydraulic jump.

mesh was containing 2,968,944 triangles (i.e. about three million elements and about 27 million unknowns!). In our anisotropic simulation, the initial mesh was containing about 2000 triangles. After 10 seconds of computation, this number was about 16,000 (i.e. 2 orders of magnitude less than the equivalent uniform mesh). Figures 12, 13 and 14 show adapted meshes, water level contours and discontinuity indicator at different times. The indicator \mathcal{I}_e of Equation (11) was only shown when its value was greater than 1. The threshold value $\mathcal{I}_e = 1$ was convenient to detect the principal hydraulic jumps of the flow. Note that the indicator did not detect the slope discontinuity of the water bed at $y = \pm 1.8\text{m}$. On the other side, Figure 14 show that, even if the indicator \mathcal{I}_e did not predict any discontinuity along this line, the second order derivative of h being large there, the mesh is refined along that line because of the large value of $\partial^2 h / \partial y^2$. It is an example among many others that show that the use of second order derivatives may be an incorrect strategy for predicting the discretization error.

Another useful application of the discontinuity indicator \mathcal{I}_e is related to limiting. With limiting only used near discontinuities, i.e. when $\mathcal{I}_e > 1$, we need not be as concerned with maintaining a high order of accuracy; thus, we focus on the slope limiting procedure introduced by Cockburn and Shu [6]. Slope limiting compares solution gradients on e with average solution gradients on neighboring elements e_k . The computed and average gradients are compared and elemental slopes are restricted to the range spanned by neighboring averages when all have the same sign. Slopes are set to zero should signs disagree [6, 26, 27].

The adaptive mesh procedure was able to capture the complex features of the flow with

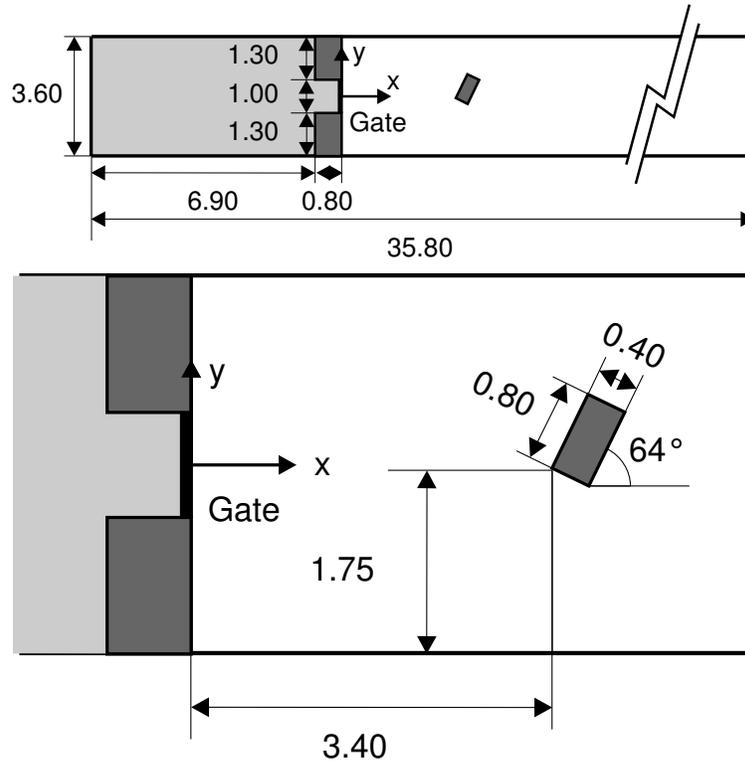


Figure 11. Experimental setup for the dam break problem in presence of a building.

a high level of accuracy. The comparison with our experiment (Figure 15) shows that the adaptive DGM was able to predict complex wave interactions with accuracy.

5. Conclusions

In this paper, we have shown the application of an adaptive meshing procedure on transient computation of shallow water equations. The adaptive procedures were able to effectively predict the wave motions and locations through a large number of mesh adaptation steps. The mesh adaptations did effectively control the discretization errors and were executed such that they did not introduce excessive numerical dissipation during the required projection operations.

The ability to perform a large number of adaptation (125 mesh adaptations for the radial dam break and more than 500 for the isolated building case) without degrading the solution is principally due to the underlying numerical schemes. The use of the Discontinuous Galerkin Method with orthogonal basis and controlled local mesh modification operations allows conservative and high order mesh-to-mesh projections, avoiding there the usual drawbacks

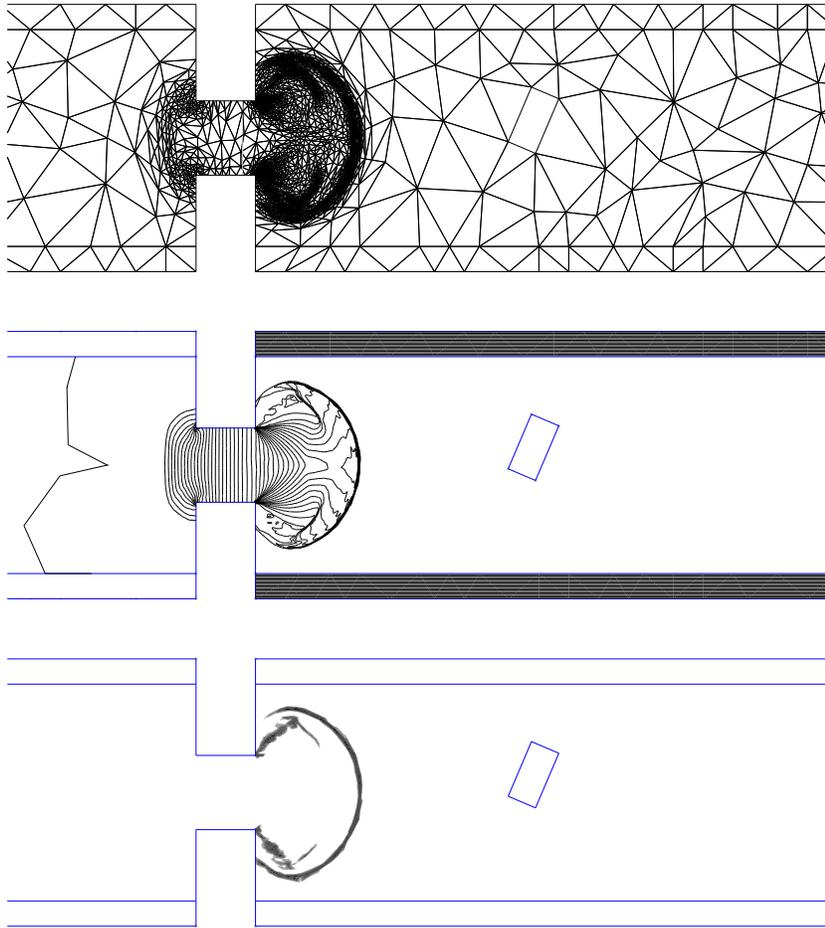


Figure 12. Adapted mesh (top), water level contours (middle) and discontinuity detector (bottom) at $t = 0.66$ s. At this time, the mesh is composed of 4,413 triangles which correspond to 39,717 unknowns.

of multiple adaptations.

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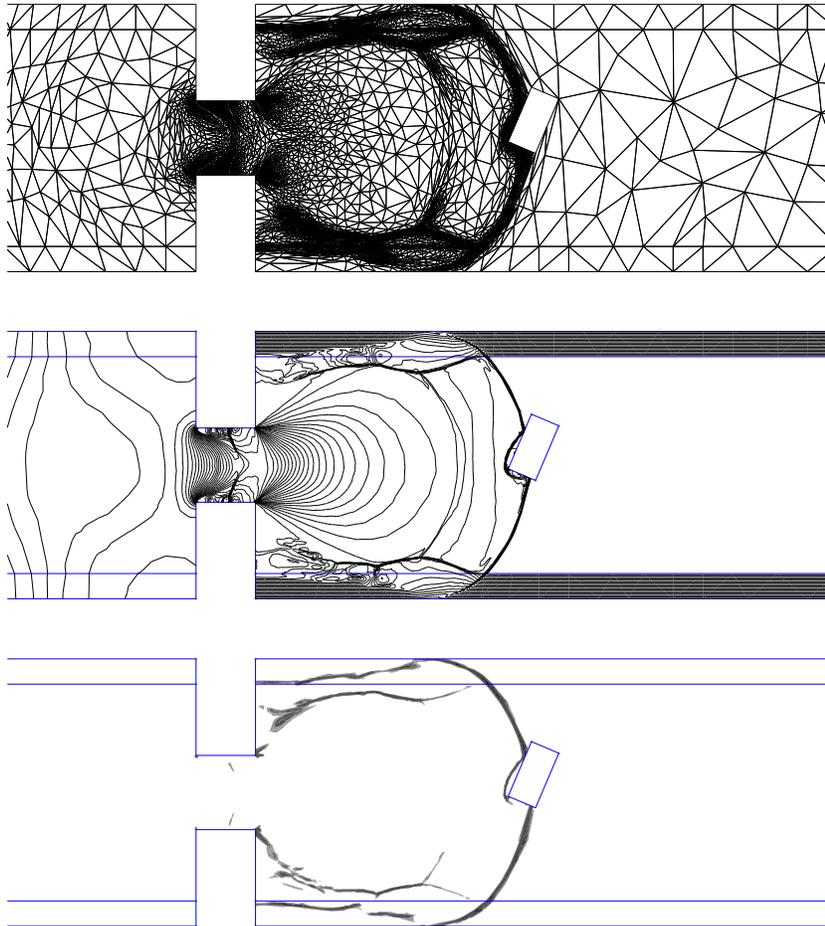


Figure 13. Adapted mesh (top), water level contours (middle) and discontinuity detector (bottom) at $t = 1.97$ s. At this time, the mesh is composed of 10,258 triangles which correspond to 92,322 unknowns.

available.

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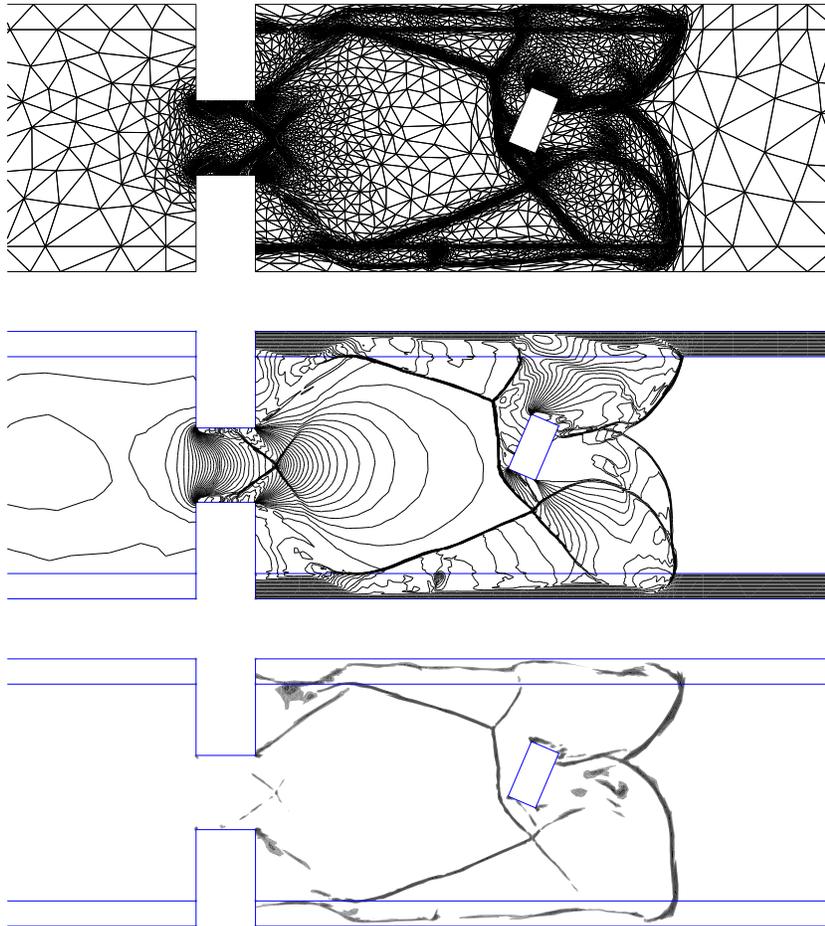


Figure 14. Adapted mesh (top), water level contours (middle) and discontinuity detector (bottom) at $t = 3.42$ s. At this time, the mesh is composed of 15,606 triangles which correspond to 140,454 unknowns.

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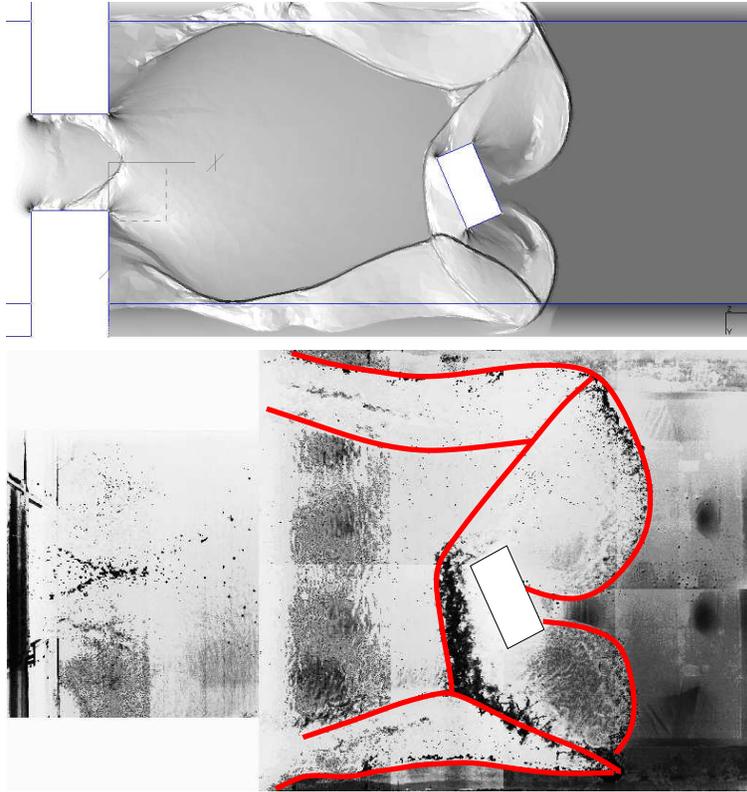


Figure 15. Water level at $t = 2.63$ s. for the isolated building problem. On top, the anisotropic computation and on bottom, the experiment. Red lines were drawn on top of the picture in order to enhance the hydraulic jumps.

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