# UNIQUENESS AND SOLUTION OF TIME-HARMONIC INVERSE VISCOELASTICITY PROBLEMS WITH INTERIOR DATA 

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## CONTENTS

LIST OF FIGURES ..... v
DEDICATION ..... ix
ACKNOWLEDGEMENT ..... x
ABSTRACT ..... xi

1. Introduction ..... 1
1.1 Elasticity Imaging/Elastography ..... 1
1.2 Inverse Problem ..... 4
1.2.1 Local Homogeneity Assumption ..... 4
1.2.2 Neglecting Pressure ..... 5
1.2.3 Plane Stress/Strain ..... 6
1.2.4 3D Incompressible Elasticity Equation ..... 7
1.3 Uniqueness of Inverse Problem ..... 8
1.4 Research Objectives ..... 9
1.5 Organization of the Thesis ..... 11
2. Anti-Plane Shear and Plane Stress Elasticity ..... 13
2.1 Problem Formulation ..... 14
2.1.1 Anti-Plane Shear ..... 15
2.1.2 Plane Stress Approximation ..... 16
2.1.3 Analysis of the Strong Form ..... 16
2.2 Weak Form: Complex Adjoint Weighted Equations ..... 17
2.2.1 Problem Formulation ..... 17
2.2.2 Analysis of CAWE Formulation ..... 18
2.3 Regularization of the CAWE Formulation ..... 21
2.3.1 Motivation for the Need for Regularization ..... 21
2.3.2 Regularized CAWE ..... 23
2.4 Numerical Approximation ..... 23
2.4.1 Synthetic Data ..... 25
2.4.2 Ultrasound Measured Data ..... 31
2.4.3 MR Measured Data ..... 33
2.5 Chapter Summary ..... 38
3. Plane Strain ..... 40
3.1 Strong Form ..... 41
3.1.1 Problem Statement ..... 41
3.1.2 Uniqueness Results ..... 42
3.1.3 Unified Equation for Multiple Loadings ..... 54
3.2 Weak Form: Complex Adjoint Weighted Equations ..... 55
3.2.1 Problem Formulation ..... 55
3.2.2 Analysis of CAWE Formulation ..... 57
3.3 Numerical Approximation ..... 60
3.3.1 Synthetic Data ..... 62
3.4 Chapter Summary ..... 73
4. Three-Dimensional Time-Harmonic Viscoelasticity Problem ..... 77
4.1 Strong Form ..... 78
4.1.1 Problem Formulation ..... 78
4.1.2 Uniqueness Results ..... 79
4.2 Weak Form: Complex Adjoint Weighted Equations ..... 88
4.2.1 Problem Formulation ..... 88
4.2.2 Analysis of CAWE Formulation ..... 89
4.3 Numerical Approximation ..... 92
4.3.1 MR Measured Data ..... 94
4.4 Chapter Summary ..... 98
5. Conclusions ..... 103
REFERENCES ..... 106
APPENDICES
A. Compatibility Conditions of Two Displacement Fields ..... 112
B. Uniqueness Proof for Hyperbolic System of Second Order ..... 114
C. Elimination of Dependence on $\mu^{i}$ ..... 120

## LIST OF FIGURES

### 2.1 Real part of the wave fields used for the inverse problem. <br> 26

2.2 Reconstruction of the shear modulus using CAWE with zero-noise dis
placement fields. Left: Real component; Right: Imaginary component.
2.3 Reconstruction of the shear modulus using LS with zero-noise displacement fields. Left: Real component; Right: Imaginary component.27
2.4 Variation of material properties along a horizontal line through the center of the inclusion with no noise and regularization. Left: Real component; Right: Imaginary component28
2.5 Reconstruction of the shear modulus using CAWE with zero-noise displacement fields $\left(\alpha_{j}=100(j=1,2)\right)$. Left: Real component; Right: Imaginary component.28
2.6 Reconstruction of the shear modulus using LS with zero-noise displacement fields $\left(\alpha_{j}=100(j=1,2)\right)$. Left: Real component; Right: Imaginary component.
2.7 Variation of material properties along a horizontal line through the center of the inclusion with no noise and $\alpha_{j}=1.0 e 2(j=1,2)$. Left: Real component; Right: Imaginary component.29
2.8 Reconstruction of the shear modulus using CAWE with noisy displacement fields $\left(\alpha_{j}=1000(j=1,2)\right)$. Left: Real component; Right: Imaginary component.30
2.9 Reconstruction of the shear modulus using LS with noisy displacement fields $\left(\alpha_{j}=1000(j=1,2)\right)$. Left: Real component; Right: Imaginary component30
2.10 Variation of material properties along a horizontal line through the center of the inclusion with $3 \%$ noise and $\alpha_{j}=1.0 e 3(j=1,2)$ Left: Real component; Right: Imaginary component.
2.11 Real part of the displacement fields in the gelatin phantom measured by ultrasound.
2.12 Reconstruction of the shear modulus for the gelatin phantom using CAWE with two displacement fields $\left(\alpha_{j}=23(j=1,2)\right)$. Left: Bmode ultrasound image; Center: Real component of the shear modulus; Right: Imaginary component of the shear modulus.
2.13 (a) Plot of the real part of the complex modulus in inclusion (in blue) and in background (in green) as a function of frequency. (b) Plot of the imaginary part of the complex modulus in inclusion (in blue) and in background (in green) as a function of frequency.
2.14 Plot of the wave speed in inclusion (in red) and in background (in blue) as a function of frequency.35
2.15 Out-of-plane component of the smooth displacement field. Left: Real component; Right: Imaginary component.36
2.16 Reconstruction of the real component of the shear modulus for the gelatin phantom using CAWE with the displacement field ( $\alpha_{1}=4000, \alpha_{2}=$ $1.0 e 6)$.
2.17 Reconstruction of the real component of the shear modulus for the gelatin phantom using CAWE with the displacement field. Left: $\alpha_{1}=$ 2000; Right: $\alpha_{1}=8000 . \alpha_{2}$ is fixed at 1.0e6.38
3.1 A construction to examine the uniqueness of the plane strain problem with two measured displacement fields. Given Cauchy data ( $\mu$ and its normal derivatives $\mu_{, n}$ ) on curve C, the equation system (3.18)-(3.19) can provide the solution in $D_{1}$, the shaded square in the figure. From the knowledge of $\mu$ and its normal derivatives on $\partial D_{1}$, the equation system (3.21)-(3.22) can provide the solution in $D_{2}$. In the figure, the characteristic curves of the equation system (3.18)-(3.19) are at $\pm 45$ degree, while the characteristic curves of the equation system (3.21)(3.22) are aligned with the $x$ and $y$ axes. Alternately using the equation systems (3.18)-(3.19) and (3.21)-(3.22) in this way, the unique solution for $\mu$ fills the plane from limited initial data.
3.2 Schematic showing two waves in homogeneous medium. The calibration region is on the left shown as the shaded region. $*$ denotes locations where $\mu$ is prescribed.65
3.3 Reconstruction of the shear modulus using CAWE with zero explicit noise displacement fields in homogeneous medium. Left: Real component; Right: Imaginary component.
3.4 Schematic showing the numerical experimental setup for the inclusion problem. The bottom of the domain is fully constrained and the two lateral sides are traction free. Two time harmonic excitations are located on the top surface (in (a)) and at the top left corner (in (b)) shown as the red arrows, to generate two waves propagating approximately vertically and diagonally, respectively. The frequency of the two excitations is 200 Hz and the amplitude is 1 mm . The calibration region is the top $1 / 8$ of the domain of interest, shown as the shaded region. * denotes locations where $\mu$ is prescribed.
3.5 Real part of the two vector displacement fields generated in Abaqus. (a) Real part of the horizontal displacement component with excitation on the top edge; (b) Real part of the horizontal displacement component with excitation at the top left corner; (c) Real part of the vertical displacement component with excitation on the top edge; (d) Real part of the vertical displacement component with excitation at the top left corner.
3.6 Reconstruction of the real part of the shear modulus using CAWE with two displacement fields. The calibration region is the top part of the domain sharing its top edge but with $1 / 8$ of its height. Boundary data for the shear modulus is imposed weakly at the four corners of the calibration region
3.7 Variation of the real part of the shear modulus along a horizontal line running through the center of the inclusion with no noise and regularization parameters $\alpha_{j}=0.0$.
3.8 Reconstructions of the real part of the shear modulus using CAWE with noisy displacement fields. (a) With no smoothing of the noisy data $(s=0)$ and regularization parameters $\alpha_{j}=0(j=1, \cdots, 6)$. $s$ is the window size of the quadratic LS filter; (b) With no smoothing ( $s=0$ ) and regularization parameters $\alpha_{j}=20$. (c) With smoothing $(s=5)$ and regularization parameters $\alpha_{j}=0$; (d) With smoothing $(s=5)$ and regularization parameters $\alpha_{j}=20$
3.9 Variation of the real part of the shear modulus along a horizontal line running through the center of the inclusion for four cases: (1) With no smoothing $(s=0)$ and regularization parameters $\alpha_{j}=0(j=1, \cdots, 6)$. $s$ is the window size of the quadratic LS filter; (2) With no smoothing $(s=0)$ and regularization parameters $\alpha_{j}=20$; (3) With smoothing $(s=5)$ and regularization parameters $\alpha_{j}=0$; (4) With smoothing $(s=5)$ and regularization parameters $\alpha_{j}=20$.
4.1 A construction to examine the uniqueness of the 3D time-harmonic viscoelasticity problem with two measured displacement fields. Given two measurements, we can obtain four equations containing $\mu_{, z}, \mu_{, x z}, \mu_{, y z}, \mu_{, z z}$, and two equations only involving $x$ - and $y$-derivatives. These two equations, when restricted to the $x y$ plane, have the same form of the PDEs for the 2D plane strain problem with two measurements considered in Section 3.1.2. According to the analysis in this section, up to eight real-valued constants are required to find a unique solution in the $x y$ plane. Then we can solve for its $z$ derivatives using Equation (4.35). Thereafter, we consider any plane parallel to the $x z$ plane. Similarly, we can obtain two equations like Equation (4.32) and Equation (4.33), but containing $x$ - and $z$-derivatives. Thus with Cauchy data known on the initial curve, in this case along $y=$ const. $\& z=0$, we can determine $\mu$ anywhere in this plane. Therefore, by translating this plane along $y$ direction we can fill up the entire space.
4.2 3D displacement data with the dimension of $196 \times 156 \times 3(y \times x \times z)$. Left: 5th imaging plane; Middle: 6th imaging plane; Right: 7th imaging plane. Top: $x$ component; Middle: $y$ component; Bottom: $z$ component. 99
4.3 Reconstruction of the real part of the shear modulus in the 5 th plane from MR measured data using CAWE ( $\alpha_{1}=1.9 e 8, \alpha_{2}=5 e 11, \alpha_{j}=$ $\left.0(j=3, \cdots, 6), \beta=1.0, \gamma_{j}=1.0 e 5\right)$.
4.4 Reconstruction of the real part of the shear modulus in the 6 th plane from MR measured data using CAWE $\left(\alpha_{1}=1.9 e 8, \alpha_{2}=5 e 11, \alpha_{j}=\right.$ $\left.0(j=3, \cdots, 6), \beta=1.0, \gamma_{j}=1.0 e 5\right)$
4.5 Reconstruction of the real part of the shear modulus in the 7 th plane from measured MR data using CAWE ( $\alpha_{1}=1.9 e 8, \alpha_{2}=5 e 11, \alpha_{j}=$ $\left.0(j=3, \cdots, 6), \beta=1.0, \gamma_{j}=1.0 e 5\right)$.
B. 1 A construction to show that in the hyperbolic system of second order (B.1) and (B.2), if $u, v$ and their derivatives $u_{, x}, u_{, y}, v_{, x}, v_{, y}$ vanish on $A B$, then $u, v$ vanish identically in the entire region $\Gamma . A B$ is an initial curve. $P A$ and $P B$ are the characteristic lines.

## DEDICATION

I dedicate this thesis to my parents, without whose nurturing and encouragement, this work would have never been possible. I cannot imagine successfully finishing the mammoth task of doctoral work without their teachings and kindest love.

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#### Abstract


In elastography, the displacement field in the interior of tissue in response to an excitation is measured using either ultrasound or magnetic resonance imaging (MRI). The research is focused on solving the subsequent inverse problem of determining the spatial distribution of the viscoelastic parameters of the tissue given the knowledge of the displacement fields in its interior. In particular, the goal is to create maps of the complex-valued shear modulus for an incompressible linear viscoelastic material undergoing infinitesimal time-harmonic deformation. This problem is motivated by applications in biomechanical imaging, where the material modulus distributions are used to detect and/or diagnose cancerous tumors.

Our approach to analyzing and solving the inverse viscoelastic problem is based on recognizing that the measured displacement fields and the reconstructed material properties satisfy the appropriate equations of motion. In the most general case these are the equations of conservation of momentum for the time-harmonic response of an incompressible, isotropic material in three dimensions. Often, approximations are introduced leading to simplified models that include the scalar Helmholtz equation, anti-plane shear, plane stress and plane strain. We consider each of these models as well as the original three dimensional time-harmonic viscoelastic equations. In each case we analyze the uniqueness of the inverse problem given single or multiple measurements. We also develop and implement a unified variational method for solving all these problems.

With regards to the uniqueness of these problems we make the following observations: (1) the problem of plane stress with a single measurement is identical to that of anti-plane shear with two measurements; (2) the problem of plane strain and the 3 D problem share the same uniqueness properties, and that these problems are more ill-posed than those of plane stress and anti-plane strain; (3) in every case, including more measurements helps considerably in reducing the space of possible solutions, thus makes the solution to the problem closer to being unique.

We propose a unified direct variational approach to solve these inverse prob-
lems. This approach can accommodate multiple measurements and multiple unknowns (the shear modulus and the pressure) simultaneously with relative case. It is derived by weighting the original partial differential equation for the shear modulus by the adjoint operator acting on the complex-conjugate of the weighting functions. For this reason we refer to it as the complex adjoint weighted equation (CAWE). We consider a straightforward finite element discretization of these equations and test its performance with synthetically generated and experimentally measured data. We also append to the CAWE formulation the total variation diminishing regularization to improve its performance in the presence of noise. We conclude that the CAWE method is accurate and represents a viable approach for determining the viscoelastic properties of tissue.

## CHAPTER 1

## Introduction

For centuries, it has been known that diseases cause local mechanical property changes in soft tissues. Although the detailed mechanism of the change varies from disease to disease. In particular it has been observed that cancerous tumors tend to be stiffer than their local surroundings. Some researchers claim that this might be due to the recruitment of collagen near the tumor, or it might be because of the exudation of fluids from the vascular system into the extra- and intracellular space or by a loss of the lymphatic system [1]. Palpation is a conventional diagnostic tool to detect tumors in soft tissues, and it is based on the fact that the stiffness between the tumor and its surrounding tissues is different. However, palpation is not quantitative and cannot detect tumors if they are small or inaccessible if they are located far from the surface.

### 1.1 Elasticity Imaging/Elastography

Elastography, a novel non-invasive imaging technique, images the stiffness or strain of soft tissue in organs like the breast, kidney, liver, and prostate to detect tumors. Typically, the tissue is deformed by a mechanical stimulus, like compression, vibration or acoustic radiation force, and its displacement is measured in some way. The displacement field is then used to recover the mechanical properties, or to simply create images of the strain.

Elastography may be categorized by three criteria: the imaging technique used to measure displacements, the type of excitation and the location of the source of excitation. Typical imaging techniques used to measure displacement include ultrasound and nuclear magnetic resonance (NMR). Ultrasound imaging has the advantages of being cheaper, faster and more portable, while MR is more sensitive, is of higher resolution [2], and is able to access deeper regions [3].

Elastography may also be classified based on the type of the excitation: quasistatic or dynamic. In quasi-static methods, the tissue is deformed slowly and the
sought parameters satisfy the quasi-static elasticity equations. In this case the inertia of the tissue is neglected. The dynamic methods, on the other hand, impose time-harmonic or transient excitation. In time-harmonic excitation, the tissue is excited by one or more time-harmonic sources, and the sought parameters satisfy the time-harmonic elasticity equations which give rise to wave-like solutions. In transient excitation [4], an external or internal, time dependent pulse creates a propagating wave in the tissue whose motion is tracked. However, even in the transient the displacement field may be resolved into its time-harmonic components via a Fourier transform. The drawback of quasi-static data is that the governing equation for the sought parameters is a homogeneous partial differential equation, and boundary conditions are required to guarantee that the solution is unique. Furthermore, the direct solution of the quasi-static inverse problems has limited success [5].

Elastography may also be categorized based on the location of the source of motion as: external or internal. For external sources, the motion of the tissue is produced by a source on the surface of the body, while for internal sources, the excitation lies inside the tissue close to the region of interest. External excitation is easy to perform, while internal excitation typically relies on acoustic radiation force (ARFI), and is possibly more accurate as it avoids the influence of the surface and the surrounding tissues. It also allows deeper regions within the tissue to be probed.

The typical methods of elastography which include a mix of imaging techniques, excitation types, and source locations described above, include:

Vibrational Sonoelastography Sonoelastography is an ultrasound imaging technique that measures the amplitude response of tissues subject to a harmonic mechanical excitation [6]. In vibrational sonoelastography, the motion of the tissue is generally produced by an external source and measured using ultrasound.

Compression Elastography This is another ultrasound-based imaging technique in which displacement in the tissue is obtained under a quasi-static external compression $[7,8]$. In this technique, radio-frequency ( RF ) ultrasound signals are imaged before and after the compression, and are used to determine the intervening displacement field. From this data either an axial strain image is obtained or
an inverse problem is solved to determine the spatial distribution of mechanical properties [7, 9].

Acoustic Radiation Force Imaging (ARFI) In this technique the tissue is vibrated internally by applying an acoustic radiation force locally, which is generated by high energy, focused ultrasound. The mechanical response of the tissue near the focus of the applied radiation force is measured using ultrasound. [10, 11, 12, 13].

Transient Elastography In transient elastography the tissue is excited using pulsed external sources or internal acoustic radiation impulses. The displacement in the tissue is measured using an ultrafast ultrasound scanner with up to $10,000 \mathrm{frames} / \mathrm{s}$ [4]. The advantage of this technique is that the shear waves, especially the wavefronts, are imaged before the steady state is reached, and thus artifacts such as standing waves are avoided.

Magnetic resonance imaging of Time-Harmonic Excitation (MRE) In magnetic resonance elastography (MRE), a phase-contrast MRI technique is used to measure the displacement in tissues in response to an external time-harmonic excitation [14, 15]. Each MRE acquisition maps only one displacement component, and three displacement components are captured by repeating the experiment in three orthogonal directions. Thereafter a rich 3D complex harmonic displacement at the driving frequency is extracted by taking Fourier transform of the displacement in time domain and is used in an inverse problem to determine the mechanical properties.

Magnetic resonance imaging of Quasi-Static Excitation In this technique, MRE is used to measure the displacement in the tissue in response to a small quasistatic compression [16], and the techniques of quasi-static elastography described above are used to create maps of the mechanical properties of the tissue.

### 1.2 Inverse Problem

In this dissertation we are concerned with quantitative elasticity imaging with multiple, time-harmonic, external or internal sources. We consider the solution of the inverse problem, where one or more components of the displacement field corresponding to a time-harmonic excitation are measured and the spatial distributions of the viscoelastic parameters are sought.

The governing equations for the linear viscoelastic, time harmonic motion in an incompressible medium are

$$
\begin{equation*}
-\nabla p+\nabla \cdot(2 \mu \boldsymbol{\epsilon})+\rho \omega^{2} \boldsymbol{u}=\mathbf{0} \tag{1.1}
\end{equation*}
$$

where $\boldsymbol{u}(\boldsymbol{x})$ is the displacement field and $\boldsymbol{\epsilon}=\frac{1}{2}\left(\nabla \boldsymbol{u}+\nabla \boldsymbol{u}^{T}\right)$ is the strain tensor. The inverse problem is to find the complex-valued shear modulus, $\mu(\boldsymbol{x})$, and the hydrostatic stress, $p(\boldsymbol{x})$, that satisfy the above equation with the displacement field $\boldsymbol{u}(\boldsymbol{x})$ given.

There has been substantial work devoted to solving the time-harmonic inverse elasticity problem. These include methods based on an assumption of local homogeneity, those based on neglecting the pressure, methods that assume plane stress or plane strain and a few methods that work with the 3D time-harmonic viscoelastic equations described above.

### 1.2.1 Local Homogeneity Assumption

Within this assumption, it is assumed that the variation in the shear modulus is small on the scale of the shear wavelength. Thus the derivatives of $\mu$ is ignored in Equation (1.1) to obtain

$$
\begin{equation*}
-\nabla p+2 \mu \nabla \cdot \boldsymbol{\epsilon}+\rho \omega^{2} \boldsymbol{u}=\mathbf{0} \tag{1.2}
\end{equation*}
$$

Curl-Based Algebraic Inversion In [17], the authors consider the local homogeneity assumption, and then eliminate the pressure in the above equation by taking the curl of the equations of motion. This yields an algebraic set of equations for the spatial distribution of the shear modulus, which allows the shear modulus to be
calculated directly and locally. However, the operation of taking the curl introduces higher order derivatives which may lead to instabilities in the presence of noise.

### 1.2.2 Neglecting Pressure

In many instances in the literature, it is assumed that the gradient of the pressure term is negligible and the term $\left(\nabla \boldsymbol{u}^{T}\right) \nabla \mu$ is ignored. The argument for neglecting the pressure is that it is associated with the bulk wave, whose wavelength is very long compared to the shear wave. With these assumptions, Equation (1.1) is reduced to the scalar Helmholtz equation for each scalar component of displacement (denoted here by $u$ ):

$$
\begin{equation*}
\nabla \cdot(\mu \nabla u)+\rho \omega^{2} u=0 . \tag{1.3}
\end{equation*}
$$

The advantage of this model is that the vector displacement is decoupled such that a single displacement component is needed in the inversion scheme. It is a useful approximation in experiments, where only the axial displacement component is measured. Several approaches have been developed to solve the scalar Helmholtz equation:

Arrival Time Algorithm This algorithm was used to estimate the shear wave speed in transient elastography and supersonic imaging [18, 19]. The main idea is to find the arrival times of the wave front with the knowledge of the displacement data, and solve the inverse Eikonal equation satisfied by the arrival times to estimate the shear wave speed.

Methods Based on Solving Equation (1.3) In [20], the authors develop and test several methods for solving the scalar Helmholtz equation (Equation (1.3)) when the shear modulus is assumed to be real-valued. These methods are based on interpreting Equation (1.3) as an equation for the shear modulus, with $u$ given. The authors then propose: (1) an efficient marching scheme to solve this problem, (2) an upwinding based method, and (3) the so-called log-elastographic method. In the $\log$-elastographic method, they work with the $\log (\mu)$ as the primary unknown, rather
than $\mu$. They do this, after recognizing that the exact solution of the inverse wave equation could have rapid exponential growth and decay, and that the proposed transformation removes some of the numerical instability that emanates from this.

The local homogeneity assumption is often invoked in the scalar Helmholtz model, which gives

$$
\begin{equation*}
\mu \nabla^{2} u+\rho \omega^{2} u=0 . \tag{1.4}
\end{equation*}
$$

This equation is named Algebraic equation and can be solved as an algebraic equation in order to determine $\mu$. Several approaches have been proposed to solve it:

Algebraic Inversion Approach In [15], three equations like Equation (1.4) (one of each displacement component) are solved algebraically, in a least-squares sense, for the single unknown $\mu$.

Local Frequency Estimation (LFE) This method, which stems from an image processing technique, estimates the local spatial frequency of the shear wave in tissues [21]. It is extended to 3D in [22].

The advantage of algebraic inversion methods is that they are fast and relatively effective in reconstructing the spatial distribution of the shear modulus, but they tend to fail at the interface of different materials. The accurate detection of interfaces, is however an important goal of elastography.

### 1.2.3 Plane Stress/Strain

From the 3D time-harmonic viscoelasticity equation (Equation (1.1)), twodimensional approximations of plane stress and plane strain can be derived. In the plane stress approximation, the object is assumed to be relatively thin in one direction, say the $z$ direction. Traction free conditions on the top and bottom of this thin sheet implies that $\sigma_{i z}=0(i=x, y, z)$. Therefore, the pressure term can be expressed as $p=2 \mu \epsilon_{z z}$ (since $\sigma_{z z}=-p+2 \mu \epsilon_{z z}$ ) and substituted in the $x$ and $y$ momentum equations. This yields two scalar Helmholtz equations. Thus the methods developed for the scalar Helmholtz equation in Section 1.2.2 are applicable
for the plane stress cases.
In the plane strain approximation, the object is assumed to be infinite in the out-of-plane direction so that the strain in this direction is zero. The equation for plane strain shares the same form as the equations of time-harmonic elasto-dynamics (Equation (1.1)) but it is restricted to two dimensions and involves only two displacement components. The presence of pressure makes these equations difficult to solve, and generally speaking these equations are as "difficult" to solve as the 3D equations. The advantage is that they only require two measured displacement components. Some of the methods developed to solve these problems are:

An Approach Based on the Finite Element Method In [23], the authors work with the discretized form of the equations of motion and treat them as the equations for the pressure and the shear modulus. Thereafter, they assume that the gradient of the pressure and the shear modulus at the domain boundaries is small, and can be neglected and obtain an overdetermined system of equations and solve it using a least-squares formulation. This approach works for 2D plane strain as well as 3D elasticity problems.

2D Log-Elastographic Inverse Algorithm In [24], the authors have considered the inverse time-harmonic elasticity problem in two dimensions and after invoking the plane strain hypothesis developed a stable numerical scheme for solving for the pressure and shear modulus. Their approach derives its stability in part from the transformation of the shear modulus, $\mu$ to its $\log$, that is $v=\log \mu$.

### 1.2.4 3D Incompressible Elasticity Equation

In some experimental setups it is possible to acquire all three components of displacements in a 3 D volume. It is then possible to treat Equation (1.1) as an equation for the unknowns, $\mu$ and $p$. Several methods have been developed to solve this problem:

An Approach Based on the Finite Element Method As stated in Section 1.2.3, in [23], the authors have developed a finite element method based approach
to solve the shear modulus and the pressure for the inverse problem of 3D incompressible elasticity. This algorithm works for the 3D elasticity problem as well as the 2D plane strain problem, since the latter may be considered as a special case of the former.

Subzone-Based Optimization Approach In contrast to the " direct" approach described above, the authors in [25] solve the inverse problem by posing it as a minimization problem and solve it iteratively by utilizing gradient based techniques. The objective function is a measure of the difference between a predicted and the measured displacement field, where the predicted field is required to satisfy the equations of motion. The spatial values of the material properties are the optimization variables, whose values are altered till an optimal match between the predicted and measured displacement fields is obtained. This approach avoids the differentiation of the noisy measured displacement fields in order to evaluate strains. However, it incurs a higher computational cost associated with solving the forward elasticity problem at each iteration. The authors control this cost through a domain decomposition technique.

### 1.3 Uniqueness of Inverse Problem

An important issue that is addressed in this dissertation is the uniqueness of the solution of the time-harmonic viscoelastic inverse problems. We note that the governing equations of the scalar Helmholtz model, the plane stress model, the plane strain model and the full-blown 3D time-harmonic viscoelasticity model can be interpreted as partial differential equations with variable coefficients for the sought shear modulus and pressure field (for plane strain and 3D). Several authors have examined the uniqueness of the solution for these problems when the shear modulus is assumed to be real-valued. However, to our knowledge no such results exists for the complex-valued case.

The authors in [26] investigate the unique identification problem of elastic parameters from time-dependent displacement field in compressible medium. They first establish the shrink and the spread argument, which says that the solution that
starts as zero in a region and satisfies both the finite propagation and the unique continuation at each time slice must be identically zero for all time. Then they establish the unique identifiability of wave speed in isotropic media for both scalar and vector displacement cases. In addition, they give counterexamples to show that a single displacement data is not enough to establish a unique solution in an anisotropic medium.

In [27], the authors consider the uniqueness and nonuniqueness issues in the incompressible elastography problem for the quasi-static and elasticity problems. They provide counterexamples to show that when a single displacement field is given, the data required to find a unique solution are impossible to obtain. They prove that given two or more displacement fields, the need for boundary data is reduced significantly. For two displacement fields, at most four arbitrary constants are needed to find a unique solution. For four displacement fields, only one arbitrary constant is needed to determine the shear modulus uniquely.

The authors in [28] investigate uniqueness problems in the compressible elastography problem. They derive the exact solutions which are valid for 2D and 3D deformations, for both quasi-static and transient deformations, and find that the exact solution contains only one arbitrary multiplicative constant.

### 1.4 Research Objectives

The work described in this thesis builds on the efforts described above. It is concerned with two important and related issues in solving the inverse time harmonic viscoelastic equations: (1) Characterization of the uniqueness of the solution and (2) Developing a unified, direct approach to solving these problems that is efficient, robust and can handle multiple measurements and unknowns easily.

Uniqueness We examine the uniqueness of the solution of time-harmonic inverse problems including the cases of anti-plane shear, plane stress, plane strain and threedimensional time-harmonic viscoelasticity problems. In each case, we allow the shear modulus to be complex-valued.

1. First, we consider the anti-plane shear case with a single displacement field. In
this case the equation reduces to the scalar Helmholtz equation with complexvalued variables. We analyze the corresponding real system of equations for the real and imaginary parts of the shear modulus, and conclude that the inverse problem with a single measured displacement field is either elliptic or hyperbolic. Thereafter we consider the anti-plane shear with two displacement fields. We conclude that we only need to specify the shear modulus at one point or its mean value over the entire domain in order to obtain a unique solution.
2. Next we examine the case of plane stress and demonstrate that with a single measured displacement field it is equivalent to the anti-plane shear case with two measured fields. Hence the conclusion drawn in item 1 holds.
3. We analyze the uniqueness of the solution to the inverse problem of 2D plane strain. In this we have to solve for the pressure and the shear modulus simultaneously. We demonstrate that the solution with a single measured displacement field is either elliptic or hyperbolic, depending on the strain field. In either case boundary data is required in order to determine a unique solution. However, with two measured displacement fields only eight real-valued constants are needed to determine the spatial distribution of the shear modulus uniquely.
4. Finally we investigate the uniqueness of the solution of the inverse problem of 3D time-harmonic viscoelasticity problem. Once again we conclude that with a single measured displacement field, the problem is either elliptic or hyperbolic depending on the strain fields. In either case boundary data is required in order to determine a unique solution. Thereafter, we analyze the solution with two measured displacement fields. In this case we conclude that a unique solution for the shear modulus may be obtained if eight pieces of data are specified.

Complex Adjoint Weighted Equation For all the cases considered in this thesis (anti-plane shear, plane stress, plane strain and three dimensions) We develop
a weak, or a variational, formulation of the original problem and examine its wellposedness. Thereafter we present and implement a straightforward finite element discretization of this formulation. Our work builds on our previous effort for the quasi-static inverse elasticity problem [29, 30], where we developed a weak formulation by weighting the original differential equation by its adjoint operating on a weighting function. For the time-harmonic viscoelastic case we note that we have to work with complex conjugate of the adjoint operator in order to retain stability. Hence we dub the new formulation the complex adjoint weighted equations (CAWE). We consider versions of this formulation that allow for multiple measured displacement fields that are necessary for uniqueness. We also allow of multiple unknowns, the shear modulus and the pressure fields that are observed in the plane strain and three-dimensional cases.

### 1.5 Organization of the Thesis

The layout of the remainder of this thesis is as follows. In Chapter 2 we consider the cases of anti-plane shear and plane stress. We examine the uniqueness of the inverse problem and introduce the CAWE formulation for solving these problems. We test the performance of the CAWE method on synthetically generated and experimentally acquired displacement data and draw conclusions about its performance.

In Chapter 3, we consider the plane strain problem. This problem is complicated by the presence of pressure that cannot be easily eliminated from the equations. We examine the uniqueness of this problem and conclude that in contrast to the plane stress case, it requires at least two measured displacement fields and some additional data for a unique solution. We extend the CAWE formulation to tackle multiple measurements and unknowns and apply it to this problem. We test the performance of the resulting algorithm on synthetically generated data.

In Chapter 4, we consider the time-harmonic viscoelasticity equations in three dimensions. We conclude that its uniqueness properties are quite similar to the plan strain case. We also apply the CAWE method for solving this problem and test its performance on experimental data.

We end with conclusions for this work in Chapter 5.

## CHAPTER 2

## Anti-Plane Shear and Plane Stress Elasticity

In this chapter we develop a CAWE formulation for two special cases of twodimensional viscoelasticity not considered elsewhere, named the anti-plane shear and the plane stress conditions as a starting point of the hierarchy of the CAWE formulation. In these two conditions the pressure term is either ignored or eliminated, and finally two independent scalar Helmholtz equations can be used to describe the anti-plane shear with two measurements and the plane stress with one measurement. We analyze the well-posedness of the strong form of the problem and conclude that the existence of the solution depends on rather restrictive compatibility conditions on measured data, which is difficult to hold for noisy data. We solve this issue by developing a weak, or a variational, formulation of the original partial differential equation (PDE). This formulation is obtained by weighting the original PDE for the shear modulus by its adjoint operator acting on the complex-conjugate of the weighting functions. We term it the complex adjoint weighted equation (CAWE). We prove that the existence and uniqueness of the solution to the CAWE formulation rely on much milder conditions. We find that the Galerkin discretization of the CAWE formulation naturally leads to a stable, robust method. Finally we test the CAWE formulation through synthetically generated data and experimentally, either ultrasound-measured or magnetic resonance (MR)-measured data. We consider appending the CAWE formulation the total variation diminishing (TVD) to improve its performance.

The layout of the reminder of this chapter is as follows. In Section 2.1, we present the strong form of the problem we wish to solve and analyze its wellposedness. In Section 2.2, we present the complex adjoint weighted equations (CAWE) and analyze their properties. In Section 2.3 we present a regularized version of this algorithm. Thereafter in Section 2.4 we consider a straightforward,

[^0]finite element based approximation of these equations and present numerical results on synthetic (computer-generated) and experimentally measured data that demonstrate the performance of the method. We end with conclusions in Section 2.5.

### 2.1 Problem Formulation

At relatively low frequencies (less than 1 kHz .) many soft tissue may be modeled as linear, incompressible, isotropic viscoelastic materials. For time-harmonic excitation with frequency $\omega$ the equations of motion are then given by:

$$
\begin{align*}
\nabla \cdot \boldsymbol{\sigma}+\rho \omega^{2} \boldsymbol{u} & =\mathbf{0}  \tag{2.1}\\
\nabla \cdot \boldsymbol{u} & =0 \tag{2.2}
\end{align*}
$$

along with the constitutive equation

$$
\begin{equation*}
\boldsymbol{\sigma}=-p \mathbf{1}+2 \mu \boldsymbol{\epsilon} . \tag{2.3}
\end{equation*}
$$

Here $\boldsymbol{u}(\boldsymbol{x})$ is the displacement vector, $\boldsymbol{\epsilon}(\boldsymbol{x})=\frac{1}{2}\left(\nabla \boldsymbol{u}+\nabla \boldsymbol{u}^{T}\right)$ is the infinitesimal strain tensor, $\boldsymbol{\sigma}(\boldsymbol{x})$ is the stress tensor, $p(\boldsymbol{x})$ is the pressure, $\rho$ is the density (assumed constant here), and $\mu(\boldsymbol{x})$ is the shear modulus. In the equations above all fields, except the constants $\rho$ and $\omega$, are complex variables. A complex, frequency dependent $\mu$ allows the material to be modeled as a viscoelastic material, where the imaginary part of $\mu$ is associated with viscous relaxation. In the forward elasticity problem the material properties $\mu$ and $\rho$, and the boundary conditions, are specified and the equations above are solved to determine the displacement vector $\boldsymbol{u}$ and the pressure $p$. In the inverse problem we are considering the displacement field is specified and the spatial distribution of the shear modulus is sought.

We consider two two-dimensional approximations of these equations. These are motivated by the fact that in several imaging scenarios the displacement field is determined on a plane and not in a volume. Thus some approximation is necessary. As shown in the following paragraphs in both cases the problem reduces to: given
$\mu\left(\boldsymbol{x}_{0}\right)=\mu_{0}$ at point $\boldsymbol{x}_{0}$ find $\mu(\boldsymbol{x})$, for $\boldsymbol{x} \in \Omega$ such that

$$
\begin{equation*}
\nabla \cdot\left(\boldsymbol{a}^{(i)} \mu\right)+f^{(i)}=0, \text { in } \Omega \quad i=1,2 . \tag{2.4}
\end{equation*}
$$

### 2.1.1 Anti-Plane Shear

Here we assume that the measurements are made in the $x y$ plane while the displacement is out-of-plane. That is $\boldsymbol{u}=u(x, y) \boldsymbol{e}_{z}$, where $\boldsymbol{e}_{z}$ is a unit vector along the $z$-direction. We also assume that the pressure and the shear modulus do not have any out-of-plane variations. With these assumptions, Equation (2.1) reduces to the scalar Helmholtz equation. That is given the measured field $u(\boldsymbol{x})$, the uniform density $\rho$ and frequency $\omega$ determine the modulus $\mu(\boldsymbol{x})$ such that

$$
\begin{equation*}
\nabla \cdot(\mu \nabla u)+\rho \omega^{2} u=0 \tag{2.5}
\end{equation*}
$$

In order to characterize this equation we consider the corresponding real system of equations obtained for $\boldsymbol{\mu}=\left[\mu^{r}, \mu^{i}\right]^{T}$. From Equation (2.5) we conclude that is given by

$$
\begin{equation*}
\boldsymbol{G}_{, x} \boldsymbol{\mu}_{, x}+\boldsymbol{G}_{, y} \boldsymbol{\mu}_{, y}+\nabla^{2}(\boldsymbol{G}) \boldsymbol{\mu}+\rho \omega^{2} \boldsymbol{u}=\mathbf{0} \tag{2.6}
\end{equation*}
$$

Here $\boldsymbol{G}=\left[\begin{array}{cc}u^{r} & -u^{i} \\ u^{i} & u^{r}\end{array}\right], \boldsymbol{u}=\left[u^{r}, u^{i}\right]^{T}$ and $\left[\nabla^{2}(\boldsymbol{G})\right]_{i j}=\nabla^{2} G_{i j}$. The type of the system of PDEs above and hence the required boundary data is determined by the form of the matrix $\boldsymbol{G}$. In particular when the characteristic equation $\operatorname{det}\left(\boldsymbol{G}_{, x}-\right.$ $\left.\tau \boldsymbol{G}_{, y}\right)=0$ permits real valued $\tau$ the system is hyperbolic. This occurs iff $u_{, x}^{r} u_{, y}^{i}=$ $u_{, y}^{r} u_{, x}^{i}$. Otherwise the system is elliptic. In most practical cases, we expect the system to be elliptic.

In the applications we are considering it is highly unlikely that data for $\mu$ will be available on the boundaries. Hence we resort to multiple measured fields in order to determine $\mu$ uniquely. We assume that we are given two measured fields $u^{(i)}, i=1,2$, and we would like to find a $\mu$ that satisfies the equations

$$
\begin{equation*}
\nabla \cdot\left(\mu \nabla u^{(i)}\right)+\rho \omega^{2} u^{(i)}=0, \quad i=1,2 . \tag{2.7}
\end{equation*}
$$

These equations may be written as Equation (2.4), where $\boldsymbol{a}^{(i)}=\nabla u^{(i)}$ and $f^{(i)}=$ $\rho \omega^{2} u^{(i)}$.

### 2.1.2 Plane Stress Approximation

The plane stress approximation is valid for objects with very small thickness (dimension in the $z$-direction), where the traction free boundary conditions at the top and bottom surfaces imply that $\sigma_{x z}=\sigma_{y z}=\sigma_{z z}=0$ is a reasonable assumption throughout. In the time-harmonic case it also implies that the inertia term in the momentum equation for the $z$-direction drops out. The zero stress conditions imply that $u_{x}=u_{x}(x, y), u_{y}=u_{y}(x, y)$ and that $p=2 \mu u_{z, z}=-2 \mu\left(u_{x, x}+u_{y, y}\right)$. Using this expression for pressure in the $x$ and $y$ momentum equations then yields Equation (2.4), where $\boldsymbol{a}^{(1)}=\left[4 u_{x, x}+2 u_{y, y}, u_{x, y}+u_{y, x}\right]^{T}$ and $\boldsymbol{a}^{(2)}=\left[u_{x, y}+u_{y, x}, 2 u_{x, x}+4 u_{y, y}\right]^{T}$ and $f^{(1)}=\rho \omega^{2} u_{x}$ and $f^{(2)}=\rho \omega^{2} u_{y}$.

### 2.1.3 Analysis of the Strong Form

For both anti-plane shear and plane stress we are lead to solving Equation (2.4) for the shear modulus. This system has a solution given by $\mu=\mu_{h}+\mu_{p}$, where

$$
\begin{align*}
\mu_{h}(\boldsymbol{x}) & =\mu_{0} \exp \left(-\int_{\boldsymbol{x}_{p}}^{\boldsymbol{x}} \boldsymbol{A}^{-1}\left(\boldsymbol{x}^{\prime}\right) \boldsymbol{a}\left(\boldsymbol{x}^{\prime}\right) \cdot d \boldsymbol{x}^{\prime}\right)  \tag{2.8}\\
\mu_{p}(\boldsymbol{x}) & =-\mu_{h}(\boldsymbol{x}) \int_{\boldsymbol{x}_{p}}^{\boldsymbol{x}} \frac{\boldsymbol{A}^{-1}\left(\boldsymbol{x}^{\prime}\right) \boldsymbol{f}\left(\boldsymbol{x}^{\prime}\right)}{\mu_{h}\left(\boldsymbol{x}^{\prime}\right)} \cdot d \boldsymbol{x}^{\prime} \tag{2.9}
\end{align*}
$$

This solution exists provided the matrix $\boldsymbol{A} \equiv \boldsymbol{a}^{(1) *} \otimes \boldsymbol{a}^{(1)}+\boldsymbol{a}^{(2) *} \otimes \boldsymbol{a}^{(2)}$ is invertible and $u^{(i)}$ satisfy the following compatibility conditions (see Appendix A).

$$
\begin{align*}
\nabla \times\left(\boldsymbol{A}^{-1} \boldsymbol{a}\right) & =\mathbf{0}  \tag{2.10}\\
\boldsymbol{C}: \nabla\left(\boldsymbol{A}^{-1} \boldsymbol{f}\right)+\left(\boldsymbol{A}^{-1} \boldsymbol{f}\right) \cdot \boldsymbol{C}\left(\boldsymbol{A}^{-1} \boldsymbol{a}\right) & =0 \tag{2.11}
\end{align*}
$$

where $\boldsymbol{a}=\boldsymbol{a}^{(1) *} \nabla \cdot \boldsymbol{a}^{(1)}+\boldsymbol{a}^{(2) *} \nabla \cdot \boldsymbol{a}^{(2)}, \boldsymbol{f}=\boldsymbol{a}^{(1) *} f^{(1)}+\boldsymbol{a}^{(2) *} f^{(2)}$ and $\boldsymbol{C}=\left[\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right]$.
The superscript * represents the complex conjugate of a complex field. In practice $u^{(i)}$ will be corrupted by noise and hence it is likely that the compatibility conditions
above will not be satisfied. Thus a single valued solution of Equation (2.4) may not exist; in other words, the value of the path integrals in Equation (2.9) may depend upon the integration path between $\boldsymbol{x}_{\boldsymbol{p}}$ and $\boldsymbol{x}$. In the section below we present a weak or a variational formulation of this problem which overcomes this difficulty by proving solutions under less restrictive conditions. It also ensures that the weak solution will be equal to the strong solution when the latter exists.

### 2.2 Weak Form: Complex Adjoint Weighted Equations

The proposed weak form is motivated by our previous work on the quasi-static elasticity problem [31]. In that case, we developed a weak or a variational form by weighting the residual of the original equations by the $L_{2}$-adjoint of the differential operator operating on a test function. For the time-harmonic case we do the same but operate on the complex-conjugate of the test function. This ensures the stability of the formulation.

### 2.2.1 Problem Formulation

In order to analyze the complex adjoint weighted equations (CAWE) formulation it is convenient to work with a zero specified mean problem instead of a point-specified problem. To this end we look for a $\hat{\mu}(\boldsymbol{x})$ such that

$$
\begin{equation*}
\nabla \cdot\left(\hat{\mu} \boldsymbol{a}^{(i)}\right)+f^{(i)}=0, \quad i=1,2, \tag{2.12}
\end{equation*}
$$

with the constraint $\frac{1}{V} \int_{\Omega} \hat{\mu} d \boldsymbol{x}=\bar{\mu}$ where $V=\operatorname{meas}(\Omega)$. Here $\bar{\mu}$ is selected such that $\hat{\mu}\left(\boldsymbol{x}_{0}\right)=\mu_{0}$, which guarantees that $\hat{\mu}(\boldsymbol{x})=\mu(\boldsymbol{x})$. The equation for specified zero-mean shear modulus $\tilde{\mu}(\boldsymbol{x})=\hat{\mu}(\boldsymbol{x})-\bar{\mu}$ is then given by

$$
\begin{equation*}
\nabla \cdot\left(\tilde{\mu} \boldsymbol{a}^{(i)}\right)+\tilde{f}^{(i)}=0, \quad i=1,2, \tag{2.13}
\end{equation*}
$$

along with the constraint $\frac{1}{V} \int_{\Omega} \tilde{\mu} d \boldsymbol{x}=0$. Here $\tilde{f}^{(i)}=f^{(i)}+\bar{\mu} \nabla \cdot \boldsymbol{a}^{(i)}$. In order to simplify notation from hereon we suppress the tilde superscript. Thus the problem
we wish to solve is: given $\boldsymbol{a}^{(i)}$ and $f^{(i)}$ find $\mu$ such that

$$
\begin{equation*}
\nabla \cdot\left(\mu \boldsymbol{a}^{(i)}\right)+f^{(i)}=0, \quad i=1,2, \text { in } \Omega \tag{2.14}
\end{equation*}
$$

and $\frac{1}{V} \int_{\Omega} \mu d \boldsymbol{x}=0$.
The complex adjoint weighted equation (CAWE) for this problem is given by: find $\mu \in \mathcal{V} \equiv\left\{v \in H^{1}(\Omega) \mid \int_{\Omega} v d \boldsymbol{x}=0\right\}$ such that

$$
\begin{equation*}
b(w, \mu)=l(w) \quad \forall w \in \mathcal{V} \tag{2.15}
\end{equation*}
$$

Here

$$
\begin{align*}
b(w, \mu) & =(\nabla w, \boldsymbol{A} \nabla \mu)+(\nabla w, \boldsymbol{a} \mu),  \tag{2.16}\\
l(w) & =-(\nabla w, \boldsymbol{f}), \tag{2.17}
\end{align*}
$$

and $(w, v)=\int_{\Omega} w^{*} u d \boldsymbol{x}$. Recall, $\boldsymbol{A} \equiv \boldsymbol{a}^{(1) *} \otimes \boldsymbol{a}^{(1)}+\boldsymbol{a}^{(2) *} \otimes \boldsymbol{a}^{(2)}, \boldsymbol{a}=\boldsymbol{a}^{(1) *} \nabla \cdot \boldsymbol{a}^{(1)}+$ $\boldsymbol{a}^{(2) *} \nabla \cdot \boldsymbol{a}^{(2)}, \boldsymbol{f}=\boldsymbol{a}^{(1) *} f^{(1)}+\boldsymbol{a}^{(2) *} f^{(2)}$.

Remark Another straightforward approach to solving Equation (2.14) is to look for the function that minimizes the residual of Equation (2.14) measured in the $L_{2}$ norm. This yields the least-squares (LS) formulation: find $\mu \in \mathcal{V}$ such that

$$
\begin{equation*}
\sum_{i=1}^{2} \operatorname{Re}\left\{\left(\nabla \cdot\left(w \boldsymbol{a}^{(i)}\right), \nabla \cdot\left(\mu \boldsymbol{a}^{(i)}\right)+f^{(i)}\right)\right\}=0 \quad \forall w \in \mathcal{V} \tag{2.18}
\end{equation*}
$$

This formulation coincides with the CAWE formulation when $\nabla \cdot \boldsymbol{a}^{(i)}=0$. In Section 2.4 we compare the performance of the LS formulation with the CAWE formulation, and in keeping with earlier observations [32], conclude that the LS formulation tends to be overly diffusive.

### 2.2.2 Analysis of CAWE Formulation

We now make assumptions on the measured data that determine the wellposedness of the CAWE formulation.
(i) We note that by construction $\boldsymbol{A}(\boldsymbol{x})$ is Hermitian positive semi-definite and thus has non-negative real eigenvalues $\gamma_{1}(\boldsymbol{x})$ and $\gamma_{2}(\boldsymbol{x})$. We further assume that these eigenvalues are positive and bounded everywhere in the domain, that is

$$
\begin{equation*}
0<\gamma_{0} \leq \gamma_{1}(\boldsymbol{x}), \gamma_{2}(\boldsymbol{x}) \leq \gamma_{\infty}<\infty \tag{2.19}
\end{equation*}
$$

(ii) Let $q^{2}(\boldsymbol{x})=\left|\nabla \cdot \boldsymbol{a}_{1}\right|^{2}+\left|\nabla \cdot \boldsymbol{a}_{2}\right|^{2}$. We assume that $q^{2}$ is bounded from above. That is

$$
\begin{equation*}
q^{2}(\boldsymbol{x}) \leq q_{0}^{2}<\infty \tag{2.20}
\end{equation*}
$$

(iii) Let $C_{P}$ be the Poincare constant for $\Omega$. That is $\|w\|^{2} \leq C\|\nabla w\|^{2}, \forall w \in$ $\mathcal{V}, \forall C>C_{P}$. We assume that the constants $\gamma_{0}$ and $q_{0}$ are such that

$$
\begin{equation*}
q_{0} \sqrt{\frac{C_{P}}{\gamma_{0}}} \leq 1 \tag{2.21}
\end{equation*}
$$

Theorem 1 When all three conditions above hold, $b(w, w)$ is coercive and the variational problem Equation (2.15) has a unique solution.

Proof. Our proof relies on the fact that for $w \in \mathcal{V}$, the $H^{1}$ semi-norm $\|\nabla w\|$ defines a norm.

We first prove the coercivity of the the bilinear form. From the definition of the bilinear form Equation (2.16)

$$
\begin{align*}
|b(w, w)| \geq & \operatorname{Re}\{b(w, w)\} \\
= & (\nabla w, \boldsymbol{A} \nabla w)+\operatorname{Re}\left\{\left(\boldsymbol{a}^{(1)} \cdot \nabla w,\left(\nabla \cdot \boldsymbol{a}^{(1)}\right) w\right)\right\} \\
& +\operatorname{Re}\left\{\left(\boldsymbol{a}^{(2)} \cdot \nabla w,\left(\nabla \cdot \boldsymbol{a}^{(2)}\right) w\right)\right\} \tag{2.22}
\end{align*}
$$

For any $\epsilon>0$

$$
\begin{equation*}
\operatorname{Re}\left\{\left(\boldsymbol{a}^{(1)} \cdot \nabla w,\left(\nabla \cdot \boldsymbol{a}^{(1)}\right) w\right)\right\} \geq-\frac{\epsilon}{2}\left\|\boldsymbol{a}^{(1)} \cdot \nabla w\right\|^{2}-\frac{1}{2 \epsilon}\left\|\left(\nabla \cdot \boldsymbol{a}^{(1)}\right) w\right\|^{2} \tag{2.23}
\end{equation*}
$$

Using this in Equation (2.22) and recalling that $q^{2}(\boldsymbol{x})=\left|\nabla \cdot \boldsymbol{a}_{1}\right|^{2}+\left|\nabla \cdot \boldsymbol{a}_{2}\right|^{2}$, we arrive at

$$
\begin{align*}
|b(w, w)| & \geq(\nabla w, \boldsymbol{A} \nabla w)\left(1-\frac{\epsilon}{2}\right)-\frac{1}{2 \epsilon}\|w q\|^{2}  \tag{2.24}\\
& \geq\|\nabla w\|^{2} \gamma_{0}\left(1-\frac{\epsilon}{2}\right)-\|w\|^{2} \frac{q_{0}^{2}}{2 \epsilon}  \tag{2.25}\\
& \geq\|\nabla w\|^{2} \gamma_{0}\left(1-\frac{\epsilon}{2}-\frac{q_{0}^{2} C_{P}}{2 \gamma_{0} \epsilon}\right)  \tag{2.26}\\
& \geq\|\nabla w\|^{2} \gamma_{0}\left(1-q_{0} \sqrt{\frac{C_{P}}{\gamma_{0}}}\right)  \tag{2.27}\\
& \geq C_{S}\|\nabla w\|^{2}, \tag{2.28}
\end{align*}
$$

where $C_{S}=\gamma_{0}\left(1-q_{0} \sqrt{\frac{C_{P}}{\gamma_{0}}}\right)$. In deriving this relation, in the second line we have made use of Equation (2.19) and Equation (2.20), in the third line we have used the Poincare inequality, in the fourth line we have used have set $\epsilon=q_{0} \sqrt{\frac{C_{P}}{\gamma_{0}}}$. When Equation (2.21) is satisfied, the stability parameter $C_{S}>0$ and the bilinear form is coercive.

We prove that the bilinear form is bounded as follows,

$$
\begin{align*}
|b(w, \mu)| \leq & |(\nabla w, \boldsymbol{A} \nabla \mu)|+\left|\left(\boldsymbol{a}^{(1)} \cdot \nabla w,\left(\nabla \cdot \boldsymbol{a}^{(1)}\right) \mu\right)\right| \\
& +\left|\left(\boldsymbol{a}^{(2)} \cdot \nabla w,\left(\nabla \cdot \boldsymbol{a}^{(2)}\right) \mu\right)\right|  \tag{2.29}\\
\leq & \|\nabla w\|\|\boldsymbol{A} \nabla \mu\|+\left\|\boldsymbol{a}^{(1)} \cdot \nabla w\right\|\left\|\left(\nabla \cdot \boldsymbol{a}^{(1)}\right) \mu\right\| \\
& +\left\|\boldsymbol{a}^{(2)} \cdot \nabla w\right\|\left\|\left(\nabla \cdot \boldsymbol{a}^{(2)}\right) \mu\right\|  \tag{2.30}\\
\leq & \gamma_{\infty}\|\nabla w\|\|\nabla \mu\|+2 \sqrt{\gamma_{\infty}} q_{0}\|\nabla w\|\|\mu\|  \tag{2.31}\\
\leq & C_{A}\|\nabla w\|\|\nabla \mu\|, \tag{2.32}
\end{align*}
$$

where $C_{A}=\gamma_{\infty}\left(1+\frac{2 q_{0} \sqrt{C_{P}}}{\sqrt{\gamma_{\infty}}}\right)$. In deriving this result in the second line we have used the Cauchy-Schwarz inequality, in the third line we have used Equation (2.19) and Equation (2.20), and to get to the final result we have used the Poincare inequality.

Next we prove that the linear form $l(w)$ is bounded $\forall w \in \mathcal{V}$. From the defini-
tion of $l(w)(2.17)$ we have

$$
\begin{align*}
|l(w)| & \leq \rho \omega^{2} \sum_{n=1}^{2}\left|\left(\boldsymbol{a}^{(n)} \cdot \nabla w, u^{(n)}\right)\right|+\mu_{0} \sum_{n=1}^{2}\left|\left(\boldsymbol{a}^{(n)} \cdot \nabla w, \nabla \cdot \boldsymbol{a}^{(n)}\right)\right|  \tag{2.33}\\
& \leq \rho \omega^{2} \sum_{n=1}^{2}\left\|\boldsymbol{a}^{(n)} \cdot \nabla w\right\|\left\|u^{(n)}\right\|+\mu_{0} \sum_{n=1}^{2}\left\|\boldsymbol{a}^{(n)} \cdot \nabla w\right\|\left\|\nabla \cdot \boldsymbol{a}^{(n)}\right\|  \tag{2.34}\\
& \leq\|\nabla w\| \sqrt{\gamma_{\infty}}\left(\rho \omega^{2} \sum_{n=1}^{2}\left\|u^{(n)}\right\|+\mu_{0} \sum_{n=1}^{2}\left\|\nabla \cdot \boldsymbol{a}^{(n)}\right\|\right)  \tag{2.35}\\
& \leq\|\nabla w\| 2 \sqrt{\gamma_{\infty}}\left(\rho \omega^{2} \sqrt{C_{P} \gamma_{\infty} V}+\mu_{0} q_{0} V\right) . \tag{2.36}
\end{align*}
$$

Recall that $V=\operatorname{meas}(\Omega)$. In the second line of the equation above we have made use of the Cauchy-Schwarz inequality, to get to the third line we have used Equation (2.19), and to get to the final result we have made use of Equation (2.20) and recognized that $\left\|u^{(n)}\right\|^{2} \leq C_{P}\left\|\nabla u^{(n)}\right\|^{2}=C_{P}\left\|\boldsymbol{a}^{(n)}\right\|^{2} \leq C_{P} \gamma_{\infty} V$.

Thus $b(w, \mu)$ is a bounded, coercive bilinear form, and $l(w)$ is a bounded linear form, hence from Lax-Milgram theorem (see for example [33]) the solution to Equation (2.15) exists and is unique.

Remark When only conditions 1 and 2 are satisfied, we can no longer prove that $b(w, w)$ is coercive. However by making use of Fredholm's alternative we are guaranteed the existence of a solution. When the corresponding homogeneous problem has no non-trivial solutions this solution is unique. However, when the homogeneous problem has multiple solutions, our problem too has multiple solutions.

### 2.3 Regularization of the CAWE Formulation

### 2.3.1 Motivation for the Need for Regularization

It is instructive to see when condition 3 is not satisfied, since when this occurs, the CAWE formulation looses its uniqueness. For the quasi-static case $(\omega=0)$ steep gradients in the solution $\mu$ imply large values of $\nabla \cdot \boldsymbol{a}$, since $\boldsymbol{a} \cdot \nabla \mu+\mu \nabla \cdot \boldsymbol{a}=0$. Large values of $\nabla \cdot \boldsymbol{a}$ in turn imply a large $q_{0}$, which could lead to the violation of condition 3. Thus in the quasi-static case we may lose stability near steep gradients in $\mu$. In the time-harmonic case this may happen even when $\mu$ is smooth as described below.

For propagating solutions of the Helmholtz equation $u \sim e^{i k \boldsymbol{n} \cdot \boldsymbol{x}}$, where $k$ is the wavenumber and $\boldsymbol{n}$ is the direction of propagation. This yields the estimates $|\boldsymbol{a}| \sim k$, and hence $\gamma_{0} \sim|\boldsymbol{A}| \sim k^{2}$. Further $q_{0}^{2} \sim|\nabla \cdot \boldsymbol{a}| \sim k^{4}$. In addition the Poincare constant $C_{P} \sim L^{2}$, where $L$ is the characteristic size of the domain. Using these estimates in Equation (2.21) we note that condition 3 holds when $k L \leq 1$. This indicates that the CAWE formulation may cease to be well-posed for problems at high frequencies (domains that are several multiples of the wavelength). Thus we need to regularize the CAWE formulation at large frequencies.

We may also motivate the use of regularization by analyzing the effect of noise. To do this we write the CAWE as follows

$$
\begin{equation*}
b(w, \mu ; \boldsymbol{d})=l(w ; \boldsymbol{d}) \quad \forall w \in \mathcal{V} \tag{2.37}
\end{equation*}
$$

where $\boldsymbol{d}=\left[\boldsymbol{a}^{(1)}, f^{(1)}, \boldsymbol{a}^{(2)}, f^{(2)}\right]$. In rewriting the original equation this way we are making the dependence on measured data explicit. In the case of any practical measurement the data will be tainted by noise $\delta \boldsymbol{d}$. The solution, $\mu+\delta \mu$, will satisfy

$$
\begin{equation*}
b(w, \mu+\delta \mu ; \boldsymbol{d}+\delta \boldsymbol{d})=l(w ; \boldsymbol{d}+\delta \boldsymbol{d}) \quad \forall w \in \mathcal{V} \tag{2.38}
\end{equation*}
$$

Assuming that the noise is small so that all terms that larger than $O(\delta)$ may be ignored, we use the equations above to arrive at an approximate equation for $\delta \mu$,

$$
\begin{equation*}
b(w, \delta \mu ; \boldsymbol{d})=D_{\boldsymbol{d}^{l}} l(w ; \boldsymbol{d}) \cdot \delta \boldsymbol{d}-D_{\boldsymbol{d}^{b}}(w, \mu ; \boldsymbol{d}) \cdot \delta \boldsymbol{d} \quad \forall w \in \mathcal{V} \tag{2.39}
\end{equation*}
$$

Using the stability estimate Equation (2.28) we have

$$
\begin{equation*}
C_{S}\|\nabla \delta \mu\|^{2} \leq|b(\delta \mu, \delta \mu ; \boldsymbol{d})| \leq\left|D_{\boldsymbol{d}^{\prime}} l(\delta \mu ; \boldsymbol{d}) \cdot \delta \boldsymbol{d}\right|+\left|D_{\boldsymbol{d}^{b}}(\delta \mu, \mu ; \boldsymbol{d}) \cdot \delta \boldsymbol{d}\right| . \tag{2.40}
\end{equation*}
$$

Or

$$
\begin{equation*}
\|\nabla \delta \mu\|^{2} \leq \frac{\mid D_{\boldsymbol{d}^{l}(\delta \mu ; \boldsymbol{d}) \cdot \delta \boldsymbol{d}\left|+\left|D_{\boldsymbol{d}^{b}}(\delta \mu, \mu ; \boldsymbol{d}) \cdot \delta \boldsymbol{d}\right|\right.}^{C_{S}},}{\text {. }} \tag{2.41}
\end{equation*}
$$

which indicates that $\delta \mu$ may become unbounded when $C_{S} \rightarrow 0$. This will happen
when $q_{0} \sqrt{\frac{C_{P}}{\gamma_{0}}} \rightarrow 1$, implying thereby that we need to regularize the problem in this limit.

### 2.3.2 Regularized CAWE

We regularize the CAWE formulation with total variation (TV) regularization [34, 35]. We use TV in order to preserve the sharp changes we expect to see at the interface of two different materials. We implement the TV in this form:

$$
\begin{equation*}
\mathcal{R}[\mu]=\int_{\Omega} \sqrt{|\nabla \mu|^{2}+\beta^{2}} d \boldsymbol{\Omega} \tag{2.42}
\end{equation*}
$$

Augmenting CAWE with $D_{\mu} \mathcal{R}[\mu] \cdot \omega$ leads to the following weak formulation: find $\mu \in \mathcal{V}$ such that

$$
\begin{align*}
b(w, \mu)+ & \alpha_{1} \operatorname{Re}\left(\nabla w^{r}, \frac{\nabla \mu^{r}}{\sqrt{\left|\nabla \mu^{r}\right|^{2}+\beta^{2}}}\right) \\
& +\alpha_{2} \operatorname{Re}\left(\nabla w^{i}, \frac{\nabla \mu^{i}}{\sqrt{\left|\nabla \mu^{i}\right|^{2}+\beta^{2}}}\right) \quad=l(w) \quad \forall w \in \mathcal{V} . \tag{2.43}
\end{align*}
$$

Here $\alpha_{j}(j=1,2)$ are the regularization parameters and $\beta$ is a parameter selected to ensure that the regularization term is continuous at $\nabla \mu^{r}=\mathbf{0}$ or $\nabla \mu^{i}=\mathbf{0}$. We note that the regularization term is non-linear and as a result the solution of the problem is also nonlinear.

### 2.4 Numerical Approximation

We approximate the variational problem Equation (2.15) by approximating the space of functions $\mathcal{V}$ with its finite dimensional counterpart $\mathcal{V}^{h} \subset \mathcal{V}$. For constructing $\mathcal{V}^{h}$ we use the standard piecewise constant finite element shape functions. Thus the numerical solution $\mu^{h} \approx \mu$ satisfies the following variational equation: find $\mu^{h} \in \mathcal{V}^{h}$ such that

$$
\begin{equation*}
b\left(w^{h}, \mu^{h}\right)=l\left(w^{h}\right) \quad \forall w^{h} \in \mathcal{V}^{h} \tag{2.44}
\end{equation*}
$$

Since $\mathcal{V}^{h} \subset \mathcal{V}$ the continuous solution $\mu$ also satisfies Equation (2.44). That is

$$
\begin{equation*}
b\left(w^{h}, \mu\right)=l\left(w^{h}\right) \quad \forall w^{h} \in \mathcal{V}^{h} \tag{2.45}
\end{equation*}
$$

Next we prove that our numerical solution converges at optimal rates to the exact solution under the restrictions of Section 2.2. We define the error $e=\mu-\mu^{h}$ and recognize that it is orthogonal to the finite dimensional space of weighting functions. That is subtracting Equation (2.44) from Equation (2.45) we have

$$
\begin{equation*}
b\left(w^{h}, e\right)=0 \quad \forall w^{h} \in \mathcal{V}^{h} \tag{2.46}
\end{equation*}
$$

We split the error $e=\eta+e^{h}$, where $\eta=\mu-\mu^{i}$ and $e^{h}=\mu^{i}-\mu^{h}$. Here $\mu^{i}$ is the best approximation to $\mu$ in the space $\mathcal{V}^{h}$. It could be, for example, the nodal interpolant of $\mu$. Using the stability estimate we have

$$
\begin{align*}
C_{S}\left\|\nabla e^{h}\right\|^{2} & \leq\left|b\left(e^{h}, e^{h}\right)\right| \\
& \left.\leq\left|b\left(e^{h}, e-\eta\right)\right| \quad \quad \text { (Since } e=\eta+e^{h}\right) \\
& \leq\left|b\left(e^{h}, \eta\right)\right| \quad \text { (from Equation (2.46)) } \\
& \leq C_{A}\left\|\nabla e^{h}\right\|\|\nabla \eta\| \quad \text { (from Equation (2.32)) } \tag{2.47}
\end{align*}
$$

which yields

$$
\begin{equation*}
\left\|\nabla e^{h}\right\| \leq \frac{C_{A}}{C_{S}}\|\nabla \eta\| \tag{2.48}
\end{equation*}
$$

That is the error in the finite element approximation is of the same order as the error of the best approximation.

We now test the performance of the finite element approximation of the regularized CAWE formulation on synthetically generated data and displacement measurements in tissue mimicking gels. In all cases we consider anti-plane shear case and work with a non-dimensional version of Equation (2.7) where we scale the displacements with a reference value $U_{r e f}$, the shear modulus with a reference value $\mu_{r e f}$, and distances with the representative length scale of the domain $L$. With this
non-dimensionalization these equations transform to

$$
\begin{equation*}
\nabla \cdot\left(\mu \nabla u^{(i)}\right)+k^{2} L^{2} u^{(i)}=0, \quad i=1,2, \tag{2.49}
\end{equation*}
$$

where $k=\sqrt{\rho \omega^{2} / \mu_{r e f}}$ is the wavenumber. In each case we use bilinear quadrilateral finite elements to solve the problem.

### 2.4.1 Synthetic Data

The first problem consists of a rectangular inclusion embedded in a homogeneous background. The shear modulus for the background is $\mu_{b g n d}=1+0.1 i$ and that of the inclusion is $\mu_{\text {incl }}=2.5+0.35 i$, the wavenumber $k L=30$, and the domain of the problem is a unit square. These values are selected so that the problem corresponds to a likely scenario in elasticity imaging of tissue.

We solve the forward problem of anti-plane shear using a uniform mesh of $100 \times$ 100 finite element. We model the infinite domain using the perfectly matched layers described in [36]. We consider two point sources placed at the bottom left and the top left corners (See Figure 2.1). These yield the two "measured" displacement fields $u^{(1)}$ and $u^{(2)}$. We calculate the derivatives of these fields by solving the variational problems

$$
\begin{equation*}
\left(\boldsymbol{w}^{h}, \boldsymbol{a}^{(i)}\right)=\left(\boldsymbol{w}^{h}, \nabla u^{(i)}\right), \quad i=1,2 . \tag{2.50}
\end{equation*}
$$

This yields $\boldsymbol{a}^{(i)}$ on a piecewise continuous finite element basis. In order to evaluate $\nabla \cdot \boldsymbol{a}^{(i)}$, we simply take the derivative of $\boldsymbol{a}^{(i)}$ within each element.

We use the synthetically created measured data $u^{(i)}, \boldsymbol{a}^{(i)}$ and $\nabla \cdot \boldsymbol{a}^{(i)}$ to reconstruct the shear modulus in a subset of the original domain, as indicated by the square in Figure 2.1. We work with the reduced domain so that there are no sources present in the region of reconstruction. In this figure we also indicate the extent of the inclusion with a red rectangle. We use a mesh of $40 \times 40$ elements for the inverse problem. We fix the shear modulus value at the origin to the correct value of the background, that is $1+0.1 i$. We note that even though there is no explicit noise in the data the numerical differentiation of $u^{(i)}$ introduces noise, and this effect is


Figure 2.1: Real part of the wave fields used for the inverse problem.
clearly seen in the reconstructions.
In Figure 2.2 we have shown the reconstruction using the CAWE formulation. From the plot of the real part of $\mu$ we observe that we recover the shape and the location of the inclusion well. We also recover the value of the modulus in the background and in the inclusion accurately. There are, however, some artifacts that are introduced through the noise in the derivatives of the measured data. These are in the form of wavy variations in the background and the inclusion and as overshoots at the interface. These artifacts are more obvious in the image of the imaginary part of the shear modulus, where they tend to overwhelm the entire image. We note that the amplitude of these variations is about the same for the real and the imaginary components of the shear modulus. They are seen more clearly in the latter because the absolute value of the latter is smaller. In Figure 2.4, we have plotted the variation in the reconstructed shear modulus along a horizontal line running through the center of the inclusion. This plot reaffirms the observations made in this paragraph.

In order to compare the performance of the CAWE formulation we solve the same problem using a least squares (LS) formulation. This formulation is given by Equation (2.18) in Section 2.2. The reconstructions are shown in Figure 2.3. The measured data is exactly the same as used for the CAWE formulation. We observe that LS formulation has similar artifacts and that they appear to be stronger. Also the contrast between the inclusion and the background appears to be under-


Figure 2.2: Reconstruction of the shear modulus using CAWE with zero-noise displacement fields. Left: Real component; Right: Imaginary component.
estimated and the variations within these regions (which are homogeneous) appear to be stronger. This is clearly seen in the plot of the material properties along a horizontal line through the center of the inclusion (Figure 2.4).


Figure 2.3: Reconstruction of the shear modulus using LS with zeronoise displacement fields. Left: Real component; Right: Imaginary component.

In Figures 2.5, 2.6 and 2.7 we present results for the CAWE and LS formulations using the same data but with TV regularization. The regularization parameter $\alpha_{j}=100(j=1,2)$ were the same for both cases. For the CAWE formulation we observe that the shape of the inclusion is captured accurately, and there is an error of about 0.15 units in the contrast. However, the overshoots and undershoots at the very sharp interface between the inclusion at the background persist. In com-


Figure 2.4: Variation of material properties along a horizontal line through the center of the inclusion with no noise and regularization. Left: Real component; Right: Imaginary component.
parison, the LS formulation is more inaccurate. The error in the contrast is about 0.5 units, there are variations in the background and (especially) the inclusions and there are sharp oscillations at the interface. We also note that LS solutions tend to be "diffusing" away from the sources, which are located on the left edge.


Figure 2.5: Reconstruction of the shear modulus using CAWE with zeronoise displacement fields $\left(\alpha_{j}=100(j=1,2)\right)$. Left: Real component; Right: Imaginary component.

Next we add $3 \%$ Gaussian white noise to the displacement fields and test the performance of the algorithms. The regularization parameters $\alpha_{j}=1000(j=1,2)$, and all other aspects of the reconstructions are unchanged. We remark that in eval-


Figure 2.6: Reconstruction of the shear modulus using LS with zeronoise displacement fields $\left(\alpha_{j}=100(j=1,2)\right)$. Left: Real component; Right: Imaginary component.


Figure 2.7: Variation of material properties along a horizontal line through the center of the inclusion with no noise and $\alpha_{j}=$ $1.0 e 2(j=1,2)$. Left: Real component; Right: Imaginary component.
uating the derivatives of the displacement fields we do not perform any smoothing. Instead we rely on the regularization term to provide all the necessary smoothing. The reconstruction for the CAWE formulation is shown in Figure 2.8. We observe that the shape and the location of the inclusion is recovered well, while the contrast in the real component is diminished by about $20 \%$. This is to be expected because of the higher value of the regularization parameter. We also observe the background and the inclusion now have sharp oscillations. These may be tempered somewhat, at the expense of loosing contrast, by increasing the regularization parameter. We
remark that though the reconstruction of the imaginary part of the modulus looks much poorer when compared to the real part, the magnitude of the error in both is about the same. We observe that the LS results in this case are completely incorrect (see Figure 2.9) . They tend to decay uniformly away from the left edge where the sources and the Dirichlet data for $\mu$ is specified. The comparison between the CAWE and LS reconstructions are shown in Figure 2.10.


Figure 2.8: Reconstruction of the shear modulus using CAWE with noisy displacement fields $\left(\alpha_{j}=1000(j=1,2)\right)$. Left: Real component; Right: Imaginary component.


Figure 2.9: Reconstruction of the shear modulus using LS with noisy displacement fields ( $\left.\alpha_{j}=1000(j=1,2)\right)$. Left: Real component; Right: Imaginary component.


Figure 2.10: Variation of material properties along a horizontal line through the center of the inclusion with $3 \%$ noise and $\alpha_{j}=1.0 e 3(j=1,2)$ Left: Real component; Right: Imaginary component.

### 2.4.2 Ultrasound Measured Data

In the section we apply the CAWE formulation to determine the shear modulus of tissue-mimicking gelatin phantoms. The sample consisted of a cylindrical inclusion embedded in a homogeneous background. The inclusion and the background were constructed using different gelatin concentrations in order to achieve a contrast in material properties. The details of the the experiment are described in [37].

The specimen was excited using ultrasound radiation force and the timedependent displacements were measured by cross-correlating ultrasound images. Two separate excitations were used, one close to the center of the left edge and the other on the right edge. The time-dependent excitations were Fourier-transformed in time, and the displacement field corresponding to $\omega=2 \pi \times 400 \mathrm{rad} / \mathrm{s}$ was used in the reconstructions. The density of gelatin was assumed to be $\rho=1000 \mathrm{~kg} / \mathrm{m}^{3}$, and the horizontal and vertical dimensions of the domain were $L_{x}=13.2 \mathrm{~mm}$. Since the waves are propagating horizontally, we require boundary conditions for $\mu$ on the vertical edges. There were determined by fitting a cylindrical wave in the lower, homogeneous, region of the phantom to first estimate the wave number and then the shear modulus. It was found that $\mu=39 .+i 12 k P a$ provided a good fit.

The problem was non-dimensionalized with $\mu_{\text {ref }}=39 k P a, U_{\text {ref }}=1.9482 \times$
$10^{-6} \mathrm{~m}$ and $2.1815 \times 10^{-6} \mathrm{~m}$ (for the left to right and right to left waves, respectively), $L=L_{x}=13.2 \mathrm{~mm}$. this lead to a wavenumber of $k L=7.0$. The real component of the non-dimensionalized displacement fields is show in Figure 2.11. The imaginary component is similar (with a different phase) and is not shown.



## Figure 2.11: Real part of the displacement fields in the gelatin phantom measured by ultrasound.

This boundary datum along with the measured displacements and their derivatives were used in the regularized CAWE algorithm in order to evaluate the complex shear modulus. The displacements were not smoothed and all the smoothing was handled through the TV regularization term. The reconstruction was performed on a $300 \times 23$ finite element mesh of bilinear quadrilateral elements.

Reconstruction The results are shown in Figure 2.12. We observe that we are able to see the inclusion quite clearly in the image of the real part of the shear modulus. We also observe that the inclusion material is about 2.4 times stiffer than the background. The location of the inclusion is also recovered (see the b-mode ultrasound image for calibration). The imaginary component of the shear modulus is not recovered as well. In particular the inclusion appears to be elongated in the vertical direction and compressed in the horizontal direction. This inaccuracy may be attributed to the anisotropic grid used for the measurement, which has lower resolution in the vertical direction, or it could be a result of the errors in assuming the scalar Helmholtz model for this elastic wave propagation problem.


Figure 2.12: Reconstruction of the shear modulus for the gelatin phantom using CAWE with two displacement fields ( $\alpha_{j}=23(j=$ $1,2)$ ). Left: Bmode ultrasound image; Center: Real component of the shear modulus; Right: Imaginary component of the shear modulus.

Frequency-dependence We recover the shear modulus from a range of frequency components of the displacement data. The range of the frequency is from 150 Hz to 750 Hz . In Figure 2.13(a) we plot the real part of the complex modulus in inclusion (in blue) and in background (in green) as a function of frequency. In Figure 2.13(b) we plot the imaginary part of the complex modulus in inclusion (in blue) and in background (in green) as a function of frequency. In Figure 2.14 we show the wave speed dispersion in inclusion (in red) and in background (in blue). We observe the trend that the real and imaginary parts of the complex shear modulus and the wave speed increase as frequency goes up, which is consistent with the observations in [38], [39] [40] [41].

### 2.4.3 MR Measured Data

In the section we apply the CAWE formulation to determine the shear modulus of a tissue-mimicking gelatin phantom using experiments performed at the Mayo clinic $[42,43]$. The sample consists of two cylindrical inclusions embedded in a homogeneous background. The diameters of the inclusions are 16 mm and 3 mm . The inclusions and the background were constructed using different gelatin concentrations in order to achieve a contrast in material properties. The shear modulus was estimated to be $20( \pm 3) k P a$ in the background and $130( \pm 10) k P a$ in the inclusions


Figure 2.13: (a) Plot of the real part of the complex modulus in inclusion (in blue) and in background (in green) as a function of frequency. (b) Plot of the imaginary part of the complex modulus in inclusion (in blue) and in background (in green) as a function of frequency.


## Figure 2.14: Plot of the wave speed in inclusion (in red) and in background (in blue) as a function of frequency.

using a local frequency estimation technique [42]. The details of the experiment are described in [43].

The specimen was excited using a harmonic mechanical force and the 3-D time harmonic displacements were measured by a phase-contrast MRI sequence with special cyclic motion encoding gradients [44]. The mechanical force was applied at the surface of the phantom via a contact plate which oscillates in the out-ofplane direction, parallel to the axes of the cylindrical inclusions. This configuration approximated the state of anti-plane shear discussed in Section 2.1. The excitation frequency was 300 Hz . The imaging plane consisted of $200 \times 160$ pixels of size $0.6275 \times 0.6275 \mathrm{~mm}^{2}$. Displacements were measured at eight time instances. This data was transformed to the frequency domain to obtain displacement at the driving frequency, $\omega=2 \pi \times 300 \mathrm{rad} / \mathrm{s}$. The density of gelatin was assumed to be $\rho=$ $1000 \mathrm{~kg} / \mathrm{m}^{3}$.

Based on the discussion on anti-plane shear in Section 2.1, we expect that the shear modulus satisfies an elliptic boundary value problem. Hence we require data for the shear modulus on the entire boundary of the domain of interest. The value of the shear modulus on the boundary was determined by fitting a plane wave in the lower, homogeneous, region of the phantom to first estimate the wavenumber and
then the shear modulus. It was found that $\mu=(20 .+i 0.5) k P a$ provided a good fit. This value was used as boundary data.

The measured displacement data was smoothed using a quadratic least squares filter. This filter performed a least squares fit of the displacement on to a quadratic surface (with $1, x, y, x y, x^{2} \& y^{2}$ monomials) over $4 \times 4$ window, and thus generated smooth displacements and strains.

The problem was non-dimensionalized with $\mu_{r e f}=20 k P a, U_{r e f}=1.32 \times$ $10^{-4} \mathrm{~m}$ and $L=L_{x}=0.0998 \mathrm{~m}$. This lead to a wavenumber of $k L=42$. The nondimensionalized smoothed displacement field in the out-of-plane direction is shown in Figure 2.15. In this figure we can clearly observe the scattering of the wave by the larger of the two inclusions. The effect of the smaller inclusion is not seen in this figure.


## Figure 2.15: Out-of-plane component of the smooth displacement field. Left: Real component; Right: Imaginary component.

The boundary data along with the smoothed displacement and strain data were used in the regularized CAWE algorithm in order to evaluate the complex shear modulus. The reconstruction was performed on the same mesh as the displacement measurement. Only the real part of the shear modulus was recovered since the imaginary part was much smaller in comparison.

In a typical inverse problem the regularization parameter may be determined using either Morozov's discrepancy principle or the L-curve (see [45] for example). Morozov's principle requires a precise estimate of measurement noise in an appropriate norm which is not available to us. Further we have found that the L-curve
tends not work well in conjunction with TV regularization. Instead of these we have used a-priori information in order to select the value of the parameter. In particular since we know that the background is homogeneous we have selected the smallest value of the regularization parameter which yields a roughly uniform background.

The result, obtained with the regularization parameters $\alpha_{1}=4,000, \alpha_{2}=$ $1.0 e 6$ is shown in Figure 2.16. In this figure both inclusions are seen quite clearly. The shape of the inclusions is also recovered, although a portion of the larger inclusion which is in the "shadow" of the incident wave is somewhat diminished. In this region the displacement magnitude is small, and as a result the ratio of the regularization term to the data matching term in Equation (2.43) is large. Consequently the effect of the regularization term is greater which leads to a reduced contrast between the inclusion and the background. This makes the inclusion appear incomplete. This may be overcome by selecting a regularization parameter that is proportional to the local magnitude of the data matching term and hence maintains the same ratio between the data matching and regularization terms.


Figure 2.16: Reconstruction of the real component of the shear modulus for the gelatin phantom using CAWE with the displacement field ( $\alpha_{1}=4000, \alpha_{2}=1.0 e 6$ ).

The contrast in the shear modulus between the large inclusion and the background is about 5 , while for the smaller inclusion it is around 3 . The actual value (obtained from an independent test) is around 6.5 for both. For the large inclusion this translates to an error of about $20 \%$, which may be attributed to the tendency of the TV regularization to reduce the total variation, and hence the contrast in the
image. The additional loss in contrast for the smaller inclusion may be attributed to the spatial smoothing of the displacement field. We note that we have employed a window of $4 \times 4$ pixels for this smoothing, and thus we expect it to have a significant effect on the small inclusion, which is only about 5 pixels in diameter.

It is worth noting that in this example using magnetic resonance elastography (MRE) and the CAWE method we are able to detect an inclusion as small as 3 mm in diameter. This has implications in the early detection of breast cancer. In particular in the detection of ductal carcinoma in-situ (DCIS) which is typically small in size as it is confined to a single milk duct.

The effect of varying the regularization parameter, $\alpha_{1}$, is displayed in Figure 2.17, where we have plotted reconstructions obtained with $\alpha_{1}=2,000$ and $\alpha_{1}=$ 8,000 , which correspond to half and two times, respectively, the value used in Figure 2.16. We note that with decreasing $\alpha_{1}$ the contrast in the inclusions increases. However, this also leads to spurious oscillations in the background.


Figure 2.17: Reconstruction of the real component of the shear modulus for the gelatin phantom using CAWE with the displacement field. Left: $\alpha_{1}=2000$; Right: $\alpha_{1}=8000 . \alpha_{2}$ is fixed at $1.0 e 6$.

### 2.5 Chapter Summary

We have considered the problem of determining the spatial distribution of the complex-valued shear modulus within an incompressible linear viscoelastic solid undergoing infinitesimal, time-harmonic deformation, from the knowledge of the
displacement field in its interior. We have restricted our attention to the twodimensional problems of anti-plane shear and plane stress. For both these cases (two measurements for anti-plane shear and one for plane stress) the shear modulus is required to satisfy two independent inverse Helmholtz equations. These equations permit the existence of a strong solution given that the measured data satisfy compatibility equations that are unlikely to hold for noisy measurements.

We have addressed this issue by formulating a weak, or a variational, formulation of these equations, which is obtained by weighting the original partial differential equation by its adjoint operating on the complex-conjugate of an arbitrary weighting function. We term this formulation the complex adjoint weighted equation (CAWE). We prove that these equations lead to a well-posed variational problem under less restrictive conditions on the measured data. However, at high frequency, or with rough data, they too may become ill-posed. For this reason we append to our formulation a regularization term.

We have developed a numerical method from the regularized CAWE formulation by restricting the functions spaces to standard, bilinear finite element function spaces. We have tested the performance of this method on synthetically generated data and experimentally measured data. The method successfully reconstructs real and imaginary parts of shear modulus from simulated data with $3 \%$ added noise, and further successfully reconstructs the real part of the shear modulus from measured data.

## CHAPTER 3 <br> Plane Strain

In this chapter we develop a complex adjoint-weighted equation (CAWE) formulation for inverse problems of incompressible plane strain elasticity. We first present the governing equation of the problems in strong form. We analyze the data requirements for the uniqueness of the solution to the problems. In this regard we carry out the analysis in three situations that involve

1. A single measured displacement field with boundary data available for shear modulus $\mu$ and pressure $p$;
2. A single measured displacement field with boundary data available only for the shear modulus $\mu$;
3. Two measured displacement fields with limited data available only for the shear modulus $\mu$.

This classification is based on two facts that boundary data for pressure $p$ is unlikely to be available in practical experiments, and that multiple measurements, if available, may be used to improve the stability of the inversion scheme and to reduce significantly the need for boundary data. Due to these two reasons, we focus mostly on the third case to reconstruct the spatial distribution of the complex-valued shear modulus. Nevertheless we present the analysis of the first situation for completeness, and the analysis of the second situation for experiments in which only a single displacement field is measured. For the first situation, we adopt the approach in Chapter 2, where we write the corresponding real system of equations for the real and imaginary parts of unknowns and examine the type of the problem based on the characteristic equation. To analyze the second and the third situations, we take curl of the governing equation to eliminate the pressure term and obtain the equation with only one unknown variable, the shear modulus. We find that the operation of taking curl does not change the type of the problem, and that for the first and the second situations, that is with a single measurement, the inverse plane strain
problem could be hyperbolic, parabolic or elliptic depending on the strain field. We also find that with the use of two measurements, the type of the problem is purely hyperbolic after manipulating the two equations. We conclude that at most eight real-valued constants are required to find a unique solution for the shear modulus.

Thereafter, we develop the CAWE formulation for the inverse plane strain elasticity problem. In this regard, we utilize the unified equation proposed in [31] for multiple measurements and then propose a general CAWE formulation. We perform the analysis of the general CAWE formulation and find that its properties are identical to that of AWE in [31]. The existence and uniqueness of its solution depend on relatively milder conditions on measured data compared to the strong form of the problem. Finally we test its performance with synthetically generated displacement data in homogeneous medium and smooth inclusion configurations.

The format of the remainder of this chapter is as follows. In Section 3.1 we give the problem statement and analyze its uniqueness in three situations. In Section 3.2 we describe a general CAWE formulation for multiple measurements and analyze its properties. In Section 3.3 we test the performance of the CAWE formulation on numerical examples. We end with conclusions in Section 3.4.

### 3.1 Strong Form

### 3.1.1 Problem Statement

In the category of two-dimensional assumption, an alternative assumption is plane strain. It is valid for objects that are assumed to be infinite in the out-of-plane direction, say the $z$ direction, so that the strains in this direction are zero, that is $\epsilon_{x z}=\epsilon_{y z}=\epsilon_{z z}=0$. The main difference from anti-plane shear and plane stress cases is that the pressure field cannot be eliminated algebraically and must be treated as an additional unknown. From the balance of linear momentum (Equation (2.1)) and the constitutive Equation (2.3), the inverse problem of incompressible isotropic plane strain elasticity can be stated as following: Given the measured field $\boldsymbol{u}(\boldsymbol{x})$, the density $\rho$ and the frequency $\omega$ find the shear modulus $\mu(\boldsymbol{x})$ and the pressure
$p(\boldsymbol{x})$ such that

$$
\begin{equation*}
-\nabla p+\nabla \cdot(2 \mu \boldsymbol{\epsilon})=-\rho \omega^{2} \boldsymbol{u} \tag{3.1}
\end{equation*}
$$

Here $\boldsymbol{u}(\boldsymbol{x})=\left[u_{x}(x, y), u_{y}(x, y)\right]^{T}$ is the two-dimensional vector displacement field measured in the $x y$ plane. $\boldsymbol{\epsilon}=\frac{1}{2}\left(\nabla \boldsymbol{u}+\nabla \boldsymbol{u}^{T}\right)$ is the second-order strain tensor. Here we consider the inverse problem, where the displacement field $\boldsymbol{u}$ is given and the complex-valued shear modulus $\mu$ and the pressure $p$ are sought.

In what follows, we shall examine the uniqueness of the inverse plane strain problem. In this regard we classify the type of PDE for $\mu$ and then determine the data required to find a unique solution. We will consider cases, where one or more measured displacements are known, and the spatial distribution of the complexvalued shear modulus is sought.

### 3.1.2 Uniqueness Results

We investigate the uniqueness of the problem under three situations: (1) a single measured displacement field with boundary data available for $\mu$ and $p$; (2) a single measured displacement field with boundary data available only for $\mu$; and (3) two measured displacement fields with boundary data available only for $\mu$.

Single displacement field with boundary data for $\mu$ and $p$ In order to characterize Equation (3.1) we write the corresponding real system of equations obtained for $\boldsymbol{\mu}=\left[\mu^{r}, \mu^{i}, p^{r}, p^{i}\right]^{T}$ :

$$
\begin{equation*}
\boldsymbol{A} \boldsymbol{\mu}_{, x}+\boldsymbol{B} \boldsymbol{\mu}_{, y}+\boldsymbol{C} \boldsymbol{\mu}+\rho \omega^{2} \boldsymbol{u}=\mathbf{0} . \tag{3.2}
\end{equation*}
$$

Here

$$
\begin{aligned}
\boldsymbol{A} & =\left[\begin{array}{cccc}
2 \epsilon_{x x}^{r} & -2 \epsilon_{x x}^{i} & -1 & 0 \\
2 \epsilon_{x x}^{i} & 2 \epsilon_{x x}^{r} & 0 & -1 \\
2 \epsilon_{x y}^{r} & -2 \epsilon_{x y}^{i} & 0 & 0 \\
2 \epsilon_{x y}^{i} & 2 \epsilon_{x y}^{r} & 0 & 0
\end{array}\right] \\
\boldsymbol{B} & =\left[\begin{array}{cccc}
2 \epsilon_{x y}^{r} & -2 \epsilon_{x y}^{i} & 0 & 0 \\
2 \epsilon_{x y}^{i} & 2 \epsilon_{x y}^{r} & 0 & 0 \\
-2 \epsilon_{x x}^{r} & 2 \epsilon_{x x}^{i} & -1 & 0 \\
-2 \epsilon_{x x}^{i} & -2 \epsilon_{x x}^{r} & 0 & -1
\end{array}\right] \\
\boldsymbol{C} & =\left[\begin{array}{lllll}
2\left(\epsilon_{x x, x}^{r}+\epsilon_{x y, y}^{r}\right) & -2\left(\epsilon_{x x, x}^{i}+\epsilon_{x y, y}^{i}\right) & 0 & 0 \\
2\left(\epsilon_{x x, x}^{i}+\epsilon_{x y, y}^{i}\right) & 2\left(\epsilon_{x x, x}^{r}+\epsilon_{x y, y}^{r}\right) & 0 & 0 \\
2\left(\epsilon_{x y, x}^{r}-\epsilon_{x x, y}^{r}\right) & -2\left(\epsilon_{x, x}^{i}-\epsilon_{x x, y}^{r}\right) & 0 & 0 \\
2\left(\epsilon_{x y, x}^{i}-\epsilon_{x x, y}^{i}\right) & 2\left(\epsilon_{x y, x}^{r}-\epsilon_{x x, y}^{r}\right) & 0 & 0
\end{array}\right] \\
\boldsymbol{u} & =\left[\begin{array}{llll}
u_{x}^{r} \\
u_{x}^{i} \\
u_{y}^{r} \\
u_{y}^{i}
\end{array}\right],
\end{aligned}
$$

where we have made use of the fact that $\epsilon_{x x}=-\epsilon_{y y}$, that is, the material is incompressible. The classification for the system of PDEs above and, hence the required boundary data are determined by the characteristic equation $\operatorname{det}(\boldsymbol{A}-\tau \boldsymbol{B})=0$. If this equation does not possess real roots (or called eigenvalue) $\tau$, then the problem is elliptic. If this equation possesses four real distinct roots, or the solutions to this equation are real and the system is not defective, the problem is hyperbolic. If the solutions to this equation are real, but the system is defective, the system is parabolic. Let us recall that a system of size $n$ is said non defective if its eigenvectors generate $\mathcal{R}^{n}$, that is, the algebraic and the geometric multiplicities of each eigenvalue are identical.

Since

$$
\begin{aligned}
& \operatorname{det}(\boldsymbol{A}-\tau \boldsymbol{B}) \\
& =\operatorname{det}\left[\begin{array}{cccc}
2\left(\epsilon_{x x}^{r}-\tau \epsilon_{x y}^{r}\right) & -2\left(\epsilon_{x x}^{i}-\tau \epsilon_{x y}^{i}\right) & -1 & 0 \\
2\left(\epsilon_{x x}^{i}-\tau \epsilon_{x y}^{i}\right) & 2\left(\epsilon_{x x}^{r}-\tau \epsilon_{x y}^{r}\right) & 0 & -1 \\
2\left(\epsilon_{x y}^{r}+\tau \epsilon_{x x}^{r}\right) & -2\left(\epsilon_{x y}^{i}+\tau \epsilon_{x x}^{i}\right) & \tau & 0 \\
2\left(\epsilon_{x y}^{i}+\tau \epsilon_{x x}^{i}\right) & 2\left(\epsilon_{x y}^{r}+\tau \epsilon_{x x}^{r}\right) & 0 & \tau
\end{array}\right] \\
& =4\left(\epsilon_{x y}^{r} \tau^{2}-2 \epsilon_{x x}^{r} \tau-\epsilon_{x y}^{r}\right)^{2} \\
& \quad+4\left(\epsilon_{x y}^{i} \tau^{2}-2 \epsilon_{x x}^{i} \tau-\epsilon_{x y}^{i}\right)^{2},
\end{aligned}
$$

setting $\operatorname{det}(\boldsymbol{A}-\tau \boldsymbol{B})=0$ and assuming $\tau$ is real, the above equation implies

$$
\begin{align*}
& \left(\epsilon_{x y}^{r} \tau^{2}-2 \epsilon_{x x}^{r} \tau-\epsilon_{x y}^{r}\right)^{2}=0, \quad \text { and }  \tag{3.3}\\
& \left(\epsilon_{x y}^{i} \tau^{2}-2 \epsilon_{x x}^{i} \tau-\epsilon_{x y}^{i}\right)^{2}=0 . \tag{3.4}
\end{align*}
$$

We consider three cases.

1. $\epsilon_{x y}=0$. In this case, the solution to the equation system (3.3) -(3.4) is $\tau=0$, which is of algebraic multiplicity 4 . The associated linearly independent left eigenvectors are $\left[\begin{array}{l}0 \\ 0 \\ 1 \\ 0\end{array}\right]$ and $\left[\begin{array}{l}0 \\ 0 \\ 0 \\ 1\end{array}\right]$, so the geometric multiplicity of this eigenvalue is 2 . Since the algebraic and the geometric multiplicities are not identical, this system is defective and thus is parabolic. One example is $\boldsymbol{u}=$ $e^{\frac{i k}{\sqrt{2}}(x+y)}\left[\begin{array}{c}1 \\ -1\end{array}\right]$. Its strain tensor $\boldsymbol{\epsilon}=\left[\begin{array}{cc}\epsilon_{2} & 0 \\ 0 & -\epsilon_{2}\end{array}\right]$ with $\epsilon_{2}=\frac{i k}{\sqrt{2}} e^{\frac{i k}{\sqrt{2}}(x+y)}$.
2. $\epsilon_{x y}^{r} \neq 0, \epsilon_{x y}^{i} \neq 0$ and $\epsilon_{x x}=0$. In this case, the solutions to the equation system (3.3) -(3.4) are $\tau= \pm 1$, each of which is of algebraic multiplicity 2 . The linearly independent left eigenvectors corresponding to the root $\tau=1$ are $\left[\begin{array}{l}1 \\ 0 \\ 1 \\ 0\end{array}\right]$
$\left[\begin{array}{l}0 \\ 1 \\ 0 \\ 1\end{array}\right]$. The linearly independent left eigenvectors corresponding to the root
$\tau=-1$ are $\left[\begin{array}{c}1 \\ 0 \\ -1 \\ 0\end{array}\right]$ and $\left[\begin{array}{c}0 \\ 1 \\ 0 \\ -1\end{array}\right]$. Since the algebraic and the geometric
multiplicities of each eigenvalue are identical, the system is hyperbolic. One example is $\boldsymbol{u}=e^{i k x}\left[\begin{array}{l}0 \\ 1\end{array}\right]$. Its strain tensor $\boldsymbol{\epsilon}=\left[\begin{array}{cc}0 & \epsilon_{1} \\ \epsilon_{1} & 0\end{array}\right]$ with $\epsilon_{1}=\frac{i k}{2} e^{i k x}$.
3. $\epsilon_{x y}^{r} \neq 0, \epsilon_{x y}^{i} \neq 0, \epsilon_{x x}^{r} \neq 0$, and $\epsilon_{x x}^{i} \neq 0$. Equation (3.3) implies

$$
\begin{equation*}
\tau= \pm \frac{\epsilon_{x x}^{r}}{\epsilon_{x y}^{r}} \pm\left(\left(\frac{\epsilon_{x x}^{r}}{\epsilon_{x y}^{r}}\right)^{2}+1\right)^{1 / 2} \tag{3.5}
\end{equation*}
$$

and Equation (3.4) implies

$$
\begin{equation*}
\tau= \pm \frac{\epsilon_{x x}^{i}}{\epsilon_{x y}^{i}} \pm\left(\left(\frac{\epsilon_{x x}^{i}}{\epsilon_{x y}^{i}}\right)^{2}+1\right)^{1 / 2} \tag{3.6}
\end{equation*}
$$

For these to hold simultaneously

$$
\begin{equation*}
\frac{\epsilon_{x x}^{r}}{\epsilon_{x y}^{r}}=\frac{\epsilon_{x x}^{i}}{\epsilon_{x y}^{i}} . \tag{3.7}
\end{equation*}
$$

Under this condition the equation system possesses four real distinct roots and thus is once again hyperbolic. This also happens in some select cases.

Otherwise this system does not possess real roots and thus is elliptic. For the hyperbolic system of first order, to obtain the unique solution of $\boldsymbol{\mu}=\left[\mu^{r}, \mu^{i}, p^{r}, p^{i}\right]$ we can require Dirichlet data for $\boldsymbol{\mu}$ on the in-flow boundaries, which are determined by the characteristic curves of the system [46]. While for the elliptic and parabolic systems of first order it is still an open question to determine the need for boundary data.

## Single displacement field with boundary data available only for $\mu$ Since

 we have boundary data available only for $\mu$, we take curl of Equations (3.1) to eliminate the pressure term and obtain the equation only containing $\mu$ :$$
\begin{equation*}
\epsilon_{x y}\left(\mu_{, y y}-\mu_{, x x}\right)+2 \epsilon_{x x} \mu_{, x y}+a_{1} \mu_{, x}+b_{1} \mu_{, y}+c_{1} \mu+d_{1}=0 \tag{3.8}
\end{equation*}
$$

In the equation above the coefficients $a_{j}, \cdots d_{j}$ are functions of the displacement components and their derivatives. In the following calculations we shall keep introducing new coefficients $a_{j}, \cdots g_{j}$ to simplify the forms of the equations. To characterize Equation (3.8) we consider the corresponding real system of equations for $\boldsymbol{\mu}=\left[\mu^{r}, \mu^{i}\right]^{T}$

$$
\begin{align*}
& \epsilon_{x y}^{r}\left(\mu_{, y y}^{r}-\mu_{, x x}^{r}\right)-\epsilon_{x y}^{i}\left(\mu_{, y y}^{i}-\mu_{, x x}^{i}\right)+2 \epsilon_{x x}^{r} \mu_{, x y}^{r}-2 \epsilon_{x x}^{i} \mu_{, x y}^{i}+\cdots=0  \tag{3.9}\\
& \epsilon_{x y}^{r}\left(\mu_{, y y}^{i}-\mu_{, x x}^{i}\right)+\epsilon_{x y}^{i}\left(\mu_{, y y}^{r}-\mu_{, x x}^{r}\right)+2 \epsilon_{x x}^{r} \mu_{, x y}^{i}+2 \epsilon_{x x}^{i} \mu_{, x y}^{r}+\cdots=0 . \tag{3.10}
\end{align*}
$$

In the above equation the dots denote the lower order derivatives of $\boldsymbol{\mu}$.
To analyze the classification for this equation system, we let $s_{1}=\mu_{, x}^{r}, s_{2}=$ $\mu_{, x}^{i}, s_{3}=\mu_{, y}^{r}, s_{4}=\mu_{, y}^{i}$ and obtain a first order PDE system for $\boldsymbol{s}=\left[\mu^{r}, \mu^{i}, s_{1}, s_{2}, s_{3}, s_{4}\right]^{T}$

$$
\begin{equation*}
\boldsymbol{A} \boldsymbol{s}_{, x}+\boldsymbol{B} \boldsymbol{s}_{, y}+\cdots=\mathbf{0} \tag{3.11}
\end{equation*}
$$

Here

$$
\begin{aligned}
\boldsymbol{A} & =\left[\begin{array}{cccccc}
0 & 0 & -\epsilon_{x y}^{r} & \epsilon_{x y}^{i} & 0 & 0 \\
0 & 0 & -\epsilon_{x y}^{i} & -\epsilon_{x y}^{r} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{array}\right] \\
\boldsymbol{B} & =\left[\begin{array}{llllll}
0 & 0 & 2 \epsilon_{x x}^{r} & -2 \epsilon_{x x}^{i} & \epsilon_{x y}^{r} & -\epsilon_{x y}^{i} \\
0 & 0 & 2 \epsilon_{x x}^{i} & 2 \epsilon_{x x}^{r} & \epsilon_{x y}^{i} & \epsilon_{x y}^{r} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}\right] .
\end{aligned}
$$

The classification for the system of PDEs above and hence the required boundary data is determined by the characteristic equation $\operatorname{det}(\boldsymbol{A}-\tau \boldsymbol{B})=0$. From

$$
\begin{aligned}
& \operatorname{det}(\boldsymbol{A}-\tau \boldsymbol{B}) \\
& =\operatorname{det}\left[\begin{array}{cccccc}
0 & 0 & -\epsilon_{x y}^{r}-2 \tau \epsilon_{x x}^{r} & \epsilon_{x y}^{i}+2 \tau \epsilon_{x x}^{i} & -\tau \epsilon_{x y}^{r} & \tau \epsilon_{x y}^{i} \\
0 & 0 & -\epsilon_{x y}^{i}-2 \tau \epsilon_{x x}^{i} & -\epsilon_{x y}^{r}-2 \tau \epsilon_{x x}^{r} & -\tau \epsilon_{x y}^{i} & -\tau \epsilon_{x y}^{r} \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\tau & 0 & -1 & 0 \\
0 & 0 & 0 & -\tau & 0 & -1
\end{array}\right] \\
& =\left(\epsilon_{x y}^{r} \tau^{2}-2 \epsilon_{x x}^{r} \tau-\epsilon_{x y}^{r}\right)^{2}+\left(\epsilon_{x y}^{i} \tau^{2}-2 \epsilon_{x x}^{i} \tau-\epsilon_{x y}^{i}\right)^{2}
\end{aligned}
$$

setting $\operatorname{det}(\boldsymbol{A}-\tau \boldsymbol{B})=0$ and assuming $\tau$ is real, we conclude that $\tau$ must simultaneously satisfy

$$
\begin{align*}
& \left(\epsilon_{x y}^{r} \tau^{2}-2 \epsilon_{x x}^{r} \tau-\epsilon_{x y}^{r}\right)^{2}=0, \quad \text { and }  \tag{3.12}\\
& \left(\epsilon_{x y}^{i} \tau^{2}-2 \epsilon_{x x}^{i} \tau-\epsilon_{x y}^{i}\right)^{2}=0 \tag{3.13}
\end{align*}
$$

This equation system is identical to the equation system (3.3)-(3.4) that we obtained for the problem with $\mu$ and $p$. Thus for the second situation, the classification of the PDE system is identical to that of the first situation. However, since this is a system only in terms of $\mu$, for the hyperbolic system of first order we require data for $\boldsymbol{s}=\left[\mu^{r}, \mu^{i}, s_{1}, s_{2}, s_{3}, s_{4}\right]^{T}$, that is, $\mu$ and its derivatives, on in-flow boundaries.

Two displacement fields with limited data available only for $\mu$ Here we assume that two displacement fields are given and some data (not necessarily boundary data) only for $\mu$ is available. We rewrite Equation (3.8) for each displacement field

$$
\begin{align*}
& \epsilon_{x y}^{(1)}\left(\mu_{, y y}-\mu_{, x x}\right)+2 \epsilon_{x x}^{(1)} \mu_{, x y}+a_{2} \mu_{, x}+b_{2} \mu_{, y}+c_{2} \mu+d_{2}=0  \tag{3.14}\\
& \epsilon_{x y}^{(2)}\left(\mu_{, y y}-\mu_{, x x}\right)+2 \epsilon_{x x}^{(2)} \mu_{, x y}+a_{3} \mu_{, x}+b_{3} \mu_{, y}+c_{3} \mu+d_{3}=0 . \tag{3.15}
\end{align*}
$$

The superscripts (1), (2) denote the first and the second displacement fields.
We separate the terms $\left(\mu_{, y y}-\mu_{, x x}\right)$ and $\mu_{, x y}$ by manipulating Equations (3.14) and (3.15). To do this, we eliminate the term $\mu_{, x y}$ by taking $\left(\epsilon_{x x}^{(2)} \times(3.14)-\epsilon_{x x}^{(1)} \times\right.$ $(3.15)) /\left(\epsilon_{x x}^{(2)} \times \epsilon_{x y}^{(1)}-\epsilon_{x x}^{(1)} \times \epsilon_{x y}^{(2)}\right)$, to arrive at

$$
\begin{equation*}
\mu_{, y y}-\mu_{, x x}+a_{4} \mu_{, x}+b_{4} \mu_{, y}+c_{4} \mu+d_{4}=0 \tag{3.16}
\end{equation*}
$$

Similarly, we eliminate the term $\left(\mu_{, y y}-\mu_{, x x}\right)$ by taking $\left(\epsilon_{x y}^{(2)} \times(3.14)-\epsilon_{x y}^{(1)} \times\right.$ (3.15)) $/ 2\left(\epsilon_{x y}^{(2)} \times \epsilon_{x x}^{(1)}-\epsilon_{x y}^{(1)} \times \epsilon_{x x}^{(2)}\right)$, to arrive at

$$
\begin{equation*}
\mu_{, x y}+a_{5} \mu_{, x}+b_{5} \mu_{, y}+c_{5} \mu+d_{5}=0 \tag{3.17}
\end{equation*}
$$

Note, that we have assumed that $\epsilon_{x y}^{(2)} \epsilon_{x x}^{(1)} \neq \epsilon_{x y}^{(1)} \epsilon_{x x}^{(2)}$. For the plane strain case with two displacement fields given, the equation system (3.16)-(3.17) are equivalent to the equation system (3.14)-(3.15). In the following, we will take advantage of the forms of the equation system (3.16)-(3.17) to analyze this problem.

To characterize the type of this problem and analyze the uniqueness of the solution, we first introduce $\mu=\mu^{r}+i \mu^{i}$ into Equation (3.16) and consider the
corresponding real system of equations for $\mu=\left[\mu^{r}, \mu^{i}\right]^{T}$

$$
\begin{align*}
& \mu_{, y y}^{r}-\mu_{, x x}^{r}-a_{6} \mu_{, y}^{r}-b_{6} \mu_{, x}^{r}-c_{6} \mu^{r}-d_{6} \mu_{, y}^{i}-e_{6} \mu_{, x}^{i}-f_{6} \mu^{i}-g_{6}=0  \tag{3.18}\\
& \mu_{, y y}^{i}-\mu_{, x x}^{i}-a_{7} \mu_{, y}^{r}-b_{7} \mu_{, x}^{r}-c_{7} \mu^{r}-d_{7} \mu_{, y}^{i}-e_{7} \mu_{, x}^{i}-f_{7} \mu^{i}-g_{7}=0 . \tag{3.19}
\end{align*}
$$

The solution to this hyperbolic system exists and is unique in the shaded region $D_{1}$ shown in Figure 3.1 given the Cauchy data on curve $C$ (see Appendix B). $D_{1}$ is bounded by the characteristics ( $y= \pm x+$ const.) of the equation system (3.18)(3.19) that circumscribe the initial curve $C$.


Figure 3.1: A construction to examine the uniqueness of the plane strain problem with two measured displacement fields. Given Cauchy data ( $\mu$ and its normal derivatives $\mu_{, n}$ ) on curve C, the equation system (3.18)-(3.19) can provide the solution in $D_{1}$, the shaded square in the figure. From the knowledge of $\mu$ and its normal derivatives on $\partial D_{1}$, the equation system (3.21)-(3.22) can provide the solution in $D_{2}$. In the figure, the characteristic curves of the equation system (3.18)-(3.19) are at $\pm 45$ degree, while the characteristic curves of the equation system (3.21)-(3.22) are aligned with the $x$ and $y$ axes. Alternately using the equation systems (3.18)-(3.19) and (3.21)(3.22) in this way, the unique solution for $\mu$ fills the plane from limited initial data.

Then we consider the second equation that we have, that is Equation (3.17). In the following we prove that we can determine $\mu$ uniquely in the region $D_{2}$ shown in Figure 3.1 with the known $\mu$ and its derivatives on $\partial D_{1}$ as Cauchy data for Equation (3.17). $D_{2}$ is bounded by the characteristics of Equation (3.17) that circumscribe $D_{1}$.

To this end, we introduce $\xi=\frac{1}{\sqrt{2}}(x+y)$ and $\eta=\frac{1}{\sqrt{2}}(-x+y)$ as independent variables instead of x and y , then Equation (3.17) goes over into

$$
\begin{equation*}
\mu_{, \xi \xi}-\mu_{, \eta \eta}+a_{8} \mu_{, \eta}+b_{8} \mu_{, \xi}+c_{8} \mu+d_{8}=0 . \tag{3.20}
\end{equation*}
$$

It shares the form of Equation (3.16) and may be rewritten as a real system of equations for $\mu=\left[\mu^{r}, \mu^{i}\right]^{T}$

$$
\begin{array}{r}
\mu_{, \xi \xi}^{r}-\mu_{, \eta \eta}^{r}-a_{9} \mu_{, \xi}^{r}-b_{9} \mu_{, \eta}^{r}-c_{9} \mu^{r}-d_{9} \mu_{, \xi}^{i}-e_{9} \mu_{, \eta}^{i}-f_{9} \mu^{i}-g_{9}=0(3.21) \\
\mu_{, \xi \xi}^{i}-\mu_{, \eta \eta}^{i}-a_{10} \mu_{, \xi}^{r}-b_{10} \mu_{, \eta}^{r}-c_{10} \mu^{r}-d_{10} \mu_{, \xi}^{i}-e_{10} \mu_{, \eta}^{i}-f_{10} \mu^{i}-g_{10}=0(3.22)
\end{array}
$$

Its characteristics are $\eta= \pm \xi+$ const. and in $x, y$-coordinates $x=$ const. and $y=$ const. shown as blue dashed lines in Figure 3.1. We can see that the solution to this hyperbolic system is unique in the region $D_{2}$, which is bounded by the characteristics of this equation system with Cauchy data known on $\partial D_{1}$ (see Appendix B).

Alternately using Equation (3.16) and Equation (3.17) in this way, we can determine $\mu$ uniquely in increasingly larger and larger regions. The only limit on the calculations above is that $\left(\epsilon_{x x}^{(2)} \times \epsilon_{x y}^{(1)}-\epsilon_{x x}^{(1)} \times \epsilon_{x y}^{(2)}\right)$ used to separate $\left(\mu_{y y}-\mu_{x x}\right)$ and $\mu_{x y}$ is supposed to be non-zero. For instance, two displacement fields with parallel characteristics satisfy $\epsilon_{x x}^{(2)} \times \epsilon_{x y}^{(1)}-\epsilon_{x x}^{(1)} \times \epsilon_{x y}^{(2)}=0$, and thus we cannot obtain a unique solution from these two measurements with Cauchy data given on the initial curve $C$.

We note that through the above calculations the size of the initial curve $C$ is arbitrary and that we only need few Cauchy data on $C$ to calculate $\mu$ everywhere. In the following, we prove that this Cauchy data satisfies an eighth order ordinary differential equation, and therefore up to eight real-valued constants are required to determine the Cauchy data necessary to solve the equation system (3.16)-(3.17).

To this end, we first eliminate the dependence of the complex-valued $\mu$ on the normal derivative, i.e., $y$-derivative of $\mu$ in the equations (3.16)-(3.17). We then remove the terms $\mu^{i}$ and its derivatives in the system and finally arrive at an eighth order ordinary differential equation for $\mu^{r}$. The first part of this operation, that is the elimination of the $y$-derivatives of $\mu$ has been done in [27], in which real-valued $\mu$ is considered. Briefly, the author eliminates the $y$-derivatives in the equations (3.16)-(3.17) in favor of the $x$-derivatives as follows:

Compute $\partial_{y}(3.17)-\partial_{x}(3.16)$ to find

$$
\begin{equation*}
a_{11} \partial_{x x x} \mu+b_{11} \partial_{x x} \mu+c_{11} \partial_{x} \mu+d_{11} \partial_{y} \mu+e_{11} \mu+f_{11}=0 . \tag{3.23}
\end{equation*}
$$

In simplifying Equation (3.23), the equations (3.16)-(3.17) are used to eliminate the terms $\partial_{y y} \mu$ and $\partial_{x y} \mu$. If $d_{11}=0$, we obtain a third-order ODE for $\mu$. While for a general case, in which $d_{11} \neq 0$, we shall eliminate the term $\partial_{y} \mu$. To this end, we first get an additional equation involving $\partial_{y} \mu$ by evaluating $d_{11} \times(3.17)-\partial_{x}(3.23)$ and get

$$
\begin{equation*}
a_{12} \partial_{x x x x} \mu+b_{12} \partial_{x x x} \mu+c_{12} \partial_{x x} \mu+d_{12} \partial_{x} \mu+e_{12} \partial_{y} \mu+f_{12} \mu+g_{12}=0 . \tag{3.24}
\end{equation*}
$$

Finally, eliminate $\partial_{y} \mu$ by computing $e_{12} \times(3.23)-d_{11} \times(3.24)$ and obtain

$$
\begin{equation*}
a_{13} \partial_{x x x x} \mu+b_{13} \partial_{x x x} \mu+c_{13} \partial_{x x} \mu+d_{13} \partial_{x} \mu+e_{13} \mu+f_{13}=0 \tag{3.25}
\end{equation*}
$$

This equation is now a fourth order ordinary differential equation for $\mu$. We shall get started our calculations from this equation, but with the complex-valued $\mu$. In the following, we shall remove the terms $\mu^{i}$ and its derivatives, and thus obtain an ordinary differential equation for $\mu^{r}$. The main idea is that we first introduce $\mu=\mu^{r}+i \mu^{i}$ into the above equation and obtain two equations for $\boldsymbol{\mu}=\left[\mu^{r}, \mu^{i}\right]$. We then eliminate the dependence of $\mu^{i}$ by manipulating the two equations. Finally we obtain an equation only for $\mu^{r}$ (see Appendix C):

$$
\begin{equation*}
a_{22} \partial_{x^{8}} \mu^{r}+b_{22} \partial_{x^{7}} \mu^{r}+c_{22} \partial_{x^{6}} \mu^{r}+d_{22} \partial_{x^{5}} \mu^{r}+\cdots+k_{22} \mu^{r}+l_{22}=0 \tag{3.26}
\end{equation*}
$$

which is an eighth order ordinary differential equation (ODE). We note that this equation is for a general case. For some special case, it could be a lower order ODE. For example, as $d_{11}=0$, it is a sixth-order ODE for $\mu^{r}$.

Consider Equation (C.11) along the $x$-axis $(y=0) . \mu^{r}$ can be determined by up to eight pieces of data of $\mu^{r}$. With this data specified, the solution on the $x$-axis

$$
\begin{equation*}
\mu^{r}(x, 0)=\bar{\mu}^{r}(x) \tag{3.27}
\end{equation*}
$$

is determined. We substitute it into Equation (C.9) and solve for $\mu^{i}(x, 0)$. Thus we know

$$
\begin{equation*}
\mu(x, 0)=\bar{\mu}(x) \tag{3.28}
\end{equation*}
$$

with

$$
\begin{equation*}
\bar{\mu}(x)=\bar{\mu}^{r}(x)+i \bar{\mu}^{i}(x) . \tag{3.29}
\end{equation*}
$$

Next we substitute Equation (3.28) into Equation (3.24) and then calculate

$$
\begin{equation*}
\partial_{y} \mu(x, 0)=\partial_{y} \bar{\mu}(x) \tag{3.30}
\end{equation*}
$$

Equation (3.28) and Equation (3.30) provide Cauchy data which are sufficient to solve the hyperbolic equation system (3.16)-(3.17).

We notice that the original equations (3.14)-(3.15) involve first derivatives of the pressure, so we specify a constant as the pressure value at one point to image the pressure distribution in the domain of interest. The reconstructed pressure value in the domain varies by a constant with the pressure value specified at one point changing. With this consideration, totally we need up to 8 calibration conditions for the shear modulus $\mu$ and two conditions for the pressure, one for each loading condition.

Our final result may be state as: Given two linear independent compatible displacement fields, $\boldsymbol{\epsilon}^{(1)}$ and $\boldsymbol{\epsilon}^{(2)}$, that are everywhere nonzero, and such that $\epsilon_{x x}^{(2)} \times$ $\epsilon_{x y}^{(1)} \neq \epsilon_{x x}^{(1)} \times \epsilon_{x y}^{(2)}$ is satisfied, let $M^{(j)}$ be the set of all functions $\mu$ that satisfies

Equations (3.14) and (3.15). Then,

$$
\begin{equation*}
\operatorname{Dim}\left\{M^{(1)} \cap M^{(2)}\right\} \leq 8 . \tag{3.31}
\end{equation*}
$$

In Equation (3.31), Dim stands for the dimension.

Remarks We notice that the problem with a single measured displacement field is elliptic in most cases and could be hyperbolic in some particular cases, while the equation with two displacement fields specified is hyperbolic unconditionally. In particular the system of equations (3.9)-(3.10) is hyperbolic/elliptic, while the system 3.21 (3.18)-(3.19) and (3.21)-(3.22), which is obtained from a linear combination of two systems like (3.9)-(3.10), is hyperbolic. Although this appears strange, it is true. Let's look at one simple example that demonstrates this type of conversion. We have two elliptic equations for $u$ :

$$
\begin{equation*}
u_{, x x}+u_{, y y}=0 \tag{3.32}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{3}{2} u_{, x x}+\frac{1}{2} u_{, y y}=0 . \tag{3.33}
\end{equation*}
$$

It is easy to verify that both Equation (3.32) and Equation (3.33) are elliptic.
Next we take $2 \times((3.33)-(3.32))$ to obtain a hyperbolic equation

$$
\begin{equation*}
u_{, x x}-u_{, y y}=0 . \tag{3.34}
\end{equation*}
$$

This equation and Equation (3.32) comprise another system, which is equivalent to the original system. Now we can see that the original equation system of two elliptic equations is converted to a system of one hyperbolic equation plus one elliptic equation. Hence we conclude that the type of a problem may change by taking linear combinations of equations.

### 3.1.3 Unified Equation for Multiple Loadings

Now that we have established the uniqueness of the complex plane strain inverse elasticity problem, we describe a numerical method to solve it. We use an unified equation that allows for multiple loadings so that we can present a general CAWE formulation (see [30]). To this end, we define $\boldsymbol{\mu}=\left(\mu, p^{(1)}, p^{(2)}, \cdots, p^{\left(N_{\text {loadings }}\right)}\right)$ and $N_{\text {loadings }}$ indexes the total number of the loading conditions, so that $\mu_{1}=\mu, \mu_{2}=$ $p^{(1)}, \cdots, \mu_{N_{\text {loadings }}+1}=p^{\left(N_{\text {loadings }}\right)}$. We rewrite the stress tensor for each loading condition using the following notation:

$$
\begin{equation*}
\boldsymbol{\sigma}^{(l)}(\boldsymbol{x})=\sum_{n=1}^{N_{\text {params }}} \mu_{n}(\boldsymbol{x}) \boldsymbol{A}^{(n, l)}(\boldsymbol{x}), \quad l=1, \cdots, N_{\text {loadings }} . \tag{3.35}
\end{equation*}
$$

Here $l$ indexes the loading conditions, the fields $\mu_{n}$ represent the shear modulus distribution and pressure fields, and $N_{\text {params }}=N_{\text {loadings }}+1$, is the number of parameters. For each loading, the equation of motion (3.1) may then be written as

$$
\begin{equation*}
\nabla \cdot\left(\sum_{n=1}^{N_{\text {params }}} \mu_{n} \boldsymbol{A}^{(n, l)}\right)=\boldsymbol{f}^{(l)} \tag{3.36}
\end{equation*}
$$

where $\boldsymbol{f}^{(l)}=-\rho \omega^{(l) 2} \boldsymbol{u}^{(l)}$. The $L_{2}$-adjoint of $\nabla \cdot\left(\sum_{n=1}^{N_{\text {params }}} \mu_{n} \boldsymbol{A}^{(n, l)}\right)$ is $\sum_{n=1}^{N_{\text {params }}} \boldsymbol{A}^{(n, l)} \nabla \mu_{n}$.
As an example consider $N_{\text {loadings }}=2$. Here we have one $\mu$ and two pressures $p^{(1)}$ and $p^{(2)}$, one for each loading condition. We define $\boldsymbol{\mu}=\left(\mu, p^{(1)}, p^{(2)}\right)$, so that $\mu_{1}=\mu, \mu_{2}=p^{(1)}$ and $\mu_{3}=p^{(2)}$. Then the stress tensor for two loadings $(l=1,2)$ are given by:

$$
\begin{equation*}
\boldsymbol{\sigma}^{(l)}=\mu \boldsymbol{A}^{(1, l)}+p_{1} \boldsymbol{A}^{(2, l)}+p_{2} \boldsymbol{A}^{(3, l)} \tag{3.37}
\end{equation*}
$$

where

$$
\begin{aligned}
& \boldsymbol{A}^{(1,1)}=\nabla \boldsymbol{u}^{(1)}+\left(\nabla \boldsymbol{u}^{(1)}\right)^{T} \\
& \boldsymbol{A}^{(2,1)}=-\mathbf{1} \\
& \boldsymbol{A}^{(3,1)}=\mathbf{0} \\
& \boldsymbol{A}^{(1,2)}=\nabla \boldsymbol{u}^{(2)}+\left(\nabla \boldsymbol{u}^{(2)}\right)^{T} \\
& \boldsymbol{A}^{(2,2)}=\mathbf{0} \\
& \boldsymbol{A}^{(3,2)}=-\mathbf{1} .
\end{aligned}
$$

### 3.2 Weak Form: Complex Adjoint Weighted Equations

### 3.2.1 Problem Formulation

We first apply the algorithm of the complex adjoint weighted equations (CAWE) for one loading condition. Thereafter we present the general CAWE formulation for multiple loading conditions.

For one loading, we define $\boldsymbol{\mu}=(\mu, p)$, the corresponding weighting function $\boldsymbol{w}=(w, q)$. The CAWE for this problem is given by: find $\boldsymbol{\mu} \in \mathcal{S}$ such that

$$
\begin{equation*}
b(\boldsymbol{w}, \boldsymbol{\mu})=l(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \mathcal{V} \tag{3.38}
\end{equation*}
$$

Here

$$
\left.\begin{array}{rl}
b(\boldsymbol{w}, \boldsymbol{\mu}) & =(2 \boldsymbol{\epsilon} \nabla w-\mathbf{1} \nabla q, \quad \nabla \cdot(2 \boldsymbol{\epsilon} \mu-\mathbf{1} p)
\end{array}\right)
$$

The weighting function space $\mathcal{V}$ and the trial solution space $\mathcal{S}$ are defined as

$$
\begin{align*}
\mathcal{V} & =\left\{(w, q) \mid w, q \in H^{1}(\Omega)\right\}  \tag{3.41}\\
\mathcal{S} & =\left\{(u, p) \mid u, p \in H^{1}(\Omega)\right\} \tag{3.42}
\end{align*}
$$

The space $\mathcal{V}$ and $\mathcal{S}$ differ from each other on the basis of the boundary conditions specified on $\mu$ and $p$. Since these may change depending on the problem type, they
are not specified here.
For multiple loadings, the general CAWE formulation is then given by: find $\boldsymbol{\mu} \in \mathcal{S}$ such that

$$
\begin{equation*}
b(\boldsymbol{w}, \boldsymbol{\mu})=l(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \mathcal{V} \tag{3.43}
\end{equation*}
$$

where

$$
\begin{align*}
b(\boldsymbol{w}, \boldsymbol{\mu}) & =\sum_{l=1}^{N_{\text {loadings }}} \sum_{m=1}^{N_{\text {params }}} b^{(l)}\left(w_{m}, \boldsymbol{\mu}\right)  \tag{3.44}\\
l(\boldsymbol{w}) & =\sum_{l=1}^{N_{\text {loadings }}} \sum_{m=1}^{N_{\text {params }}} l^{(l)}\left(w_{m}\right)  \tag{3.45}\\
b^{(l)}\left(w_{m}, \boldsymbol{\mu}\right) & =\left(\boldsymbol{A}^{(m, l)} \nabla w_{m}, \nabla \cdot \sum_{n=1}^{N_{\text {params }}}\left(\boldsymbol{A}^{(n, l)} \mu_{n}\right)\right)  \tag{3.46}\\
l^{(l)}\left(w_{m}\right) & =-\left(\boldsymbol{A}^{(m, l)} \nabla w_{m}, \boldsymbol{f}^{(l)}\right) \tag{3.47}
\end{align*}
$$

and

$$
\begin{equation*}
\boldsymbol{f}^{(l)}=\rho \omega^{(l) 2} \boldsymbol{u}^{(l)} \tag{3.48}
\end{equation*}
$$

For example, when $N_{\text {loadings }}=2$, we have $\boldsymbol{\mu}=\left(\mu, p^{(1)}, p^{(2)}\right)$, the corresponding weighting function $\boldsymbol{w}=\left(w, q^{(1)}, q^{(2)}\right)$. The CAWE for this problem is given by: find $\boldsymbol{\mu} \in \mathcal{S}$ such that

$$
\begin{equation*}
b(\boldsymbol{w}, \boldsymbol{\mu})=l(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \mathcal{V} \tag{3.49}
\end{equation*}
$$

Here

$$
\begin{align*}
& b(\boldsymbol{w}, \boldsymbol{\mu})=\left(2 \boldsymbol{\epsilon}^{(1)} \nabla w-\mathbf{1} \nabla q^{(1)}, \quad \nabla \cdot\left(2 \boldsymbol{\epsilon}^{(1)} \mu-\mathbf{1} p^{(1)}\right)\right)  \tag{3.50}\\
& +\left(2 \boldsymbol{\epsilon}^{(2)} \nabla w-1 \nabla q^{(2)}, \quad \nabla \cdot\left(2 \boldsymbol{\epsilon}^{(2)} \mu-1 p^{(2)}\right)\right) \\
& l(\boldsymbol{w})=-\left(2 \boldsymbol{\epsilon}^{(1)} \nabla w-\mathbf{1} \nabla q^{(1)}, \quad \rho \omega^{(1) 2} \boldsymbol{u}^{(1)}\right)  \tag{3.51}\\
& -\quad\left(2 \boldsymbol{\epsilon}^{(2)} \nabla w-1 \nabla q^{(2)}, \quad \rho \omega^{(2) 2} \boldsymbol{u}^{(2)}\right) .
\end{align*}
$$

### 3.2.2 Analysis of CAWE Formulation

In this section, we analyze the properties of the CAWE formulation. The analysis has been done in [30], but for real-valued problems. We extend these results to complex-valued problems. The idea here is to describe the conditions on the data for which the CAWE formulation is stable and leads to convergent numerical solutions.

## Assumptions on measured Data

(i) The CAWE bilinear form provides a natural norm on $\mathcal{V}$, that we call the $A$ norm. Thus we define:

$$
\begin{align*}
\|\boldsymbol{w}\|_{A}^{2} & \equiv \sum_{m=1}^{N_{\text {params }}} \sum_{n=1}^{N_{\text {params }}} \sum_{l=1}^{N_{\text {loadings }}}\left(\boldsymbol{A}^{(m, l)} \nabla w_{m}, \boldsymbol{A}^{(n, l)} \nabla w_{n}\right)  \tag{3.52}\\
& \equiv(\nabla \boldsymbol{w}, \boldsymbol{A} \nabla \boldsymbol{w}) \geqslant 0 \quad \forall \boldsymbol{w} \in \mathcal{V}, \tag{3.53}
\end{align*}
$$

where $[\nabla \boldsymbol{w}]_{m i}=\frac{\partial w_{m}}{\partial x_{i}}$, and $\boldsymbol{A}_{m n j k}=\sum_{i} \sum_{l} A_{i j}^{*(m, l)} A_{i k}^{(n, l)}$.
To call this a norm, we assume that

$$
\begin{equation*}
\|\boldsymbol{w}\|_{A}^{2}=0 \Leftrightarrow \boldsymbol{w}=\mathbf{0} \quad \text { in } \Omega . \tag{3.54}
\end{equation*}
$$

This happens when $\boldsymbol{A}$ is positive definite and $\mathcal{V}$ is such that a constant $\boldsymbol{w}$ is ruled out. We note it is not easy to postulate conditions on $\boldsymbol{A}^{(m, l)}$ and hence $\boldsymbol{u}^{(n)}$, to determine when this will be the case. We also note that when $\boldsymbol{A}$ is positive definite, the $A$-norm is analogous to an $H^{1}$ semi-norm on $\mathcal{V}$, and equivalent to the $H^{1}$ norm on $\mathcal{V}$ in many practical cases.

Here, we examine $\boldsymbol{A}$ for the plane strain problem with a single measurement. We note that we will work in a coordinate system aligned with the principal axes of strain. The strain can be written as $\boldsymbol{\epsilon}=\left[\begin{array}{cc}\epsilon_{1} & 0 \\ 0 & -\epsilon_{1}\end{array}\right]$ with complex-valued $\epsilon_{1} \cdot \boldsymbol{A}^{(1,1)}=2 \boldsymbol{\epsilon}, \boldsymbol{A}^{(2,1)}=-\mathbf{1}$. So $\boldsymbol{A}$, expressed as a matrix, is
$\left[\begin{array}{cccc}4\left|\epsilon_{1}\right|^{2} & 0 & -2 \epsilon_{1}^{*} & 0 \\ 0 & 4\left|\epsilon_{1}\right|^{2} & 0 & 2 \epsilon_{1}^{*} \\ -2 \epsilon_{1} & 0 & 1 & 0 \\ 0 & 2 \epsilon_{1} & 0 & 1\end{array}\right]$. The eigenvalues of $\boldsymbol{A}$ are $4|c|^{2}+1$ and 0 , each
of which has algebraic multiplicity 2 . Since the eigenvalue $4|c|^{2}+1$ is positive and another eigenvalue is zero, we conclude that $\boldsymbol{A}$ is semi-positive definite, but not positive-definite in this case. This is to be expected since with the single displacement measurement the solution for $\mu$ is not unique as shown in Section 3.1.
(ii) Let

$$
\begin{equation*}
q(\boldsymbol{w})=\sum_{m=1}^{N_{\text {params }}} \sum_{n=1}^{N_{\text {params }}} \sum_{l=1}^{N_{\text {loadings }}}\left(w_{m} \nabla \cdot \boldsymbol{A}^{(m, l)}, w_{n} \nabla \cdot \boldsymbol{A}^{(n, l)}\right) . \tag{3.55}
\end{equation*}
$$

We assume that there exists a constant $C_{p}^{A}<\infty$ such that

$$
\begin{equation*}
q(\boldsymbol{w}) \leqslant C\|\boldsymbol{w}\|_{A}^{2} \quad \forall C \geqslant C_{p}^{A} \tag{3.56}
\end{equation*}
$$

This is a generalization of Poincare inequality and will hold for small $\nabla \cdot \boldsymbol{A}^{(m, l)}$.
(iii) We assume the $A$-norm is bounded by the $H^{1}$ norm on $\mathcal{V}$. That is, there exists a finite, positive constant $C$ satisfying

$$
\begin{equation*}
\|\boldsymbol{w}\|_{A} \leqslant C\|\boldsymbol{w}\|_{1} . \tag{3.57}
\end{equation*}
$$

This implies that the Largest eigenvalue of $\boldsymbol{A}$ is bounded. When the data is such that all the conditions above are satisfied we can prove the coercivity of the CAWE formulation.

CAWE Stability We now examine coercivity of $b(\cdot, \cdot)$ :

$$
\begin{align*}
b(\boldsymbol{w}, \boldsymbol{w}) & =\sum_{m=1}^{N_{\text {params }}} \sum_{n=1}^{N_{\text {params }}} \sum_{l=1}^{N_{\text {loadings }}}\left(\boldsymbol{A}^{(m, l)} \nabla w_{m}, \nabla \cdot\left(\boldsymbol{A}^{(n, l)} w_{n}\right)\right) \\
& =\|\boldsymbol{w}\|_{A}^{2}+\sum_{m=1}^{N_{\text {params }}} \sum_{n=1}^{N_{\text {params }}} \sum_{l=1}^{N_{\text {loadings }}}\left(\boldsymbol{A}^{(m, l)} \nabla w_{m},\left(\nabla \cdot \boldsymbol{A}^{(n, l)}\right) w_{n}\right) \cdot( \tag{3.58}
\end{align*}
$$

For any $\epsilon>0$

$$
\begin{equation*}
\sum_{m=1}^{N_{\text {params }}} \sum_{n=1}^{N_{\text {params }}} \sum_{l=1}^{N_{\text {loadings }}}\left(\boldsymbol{A}^{(m, l)} \nabla w_{m},\left(\nabla \cdot \boldsymbol{A}^{(n, l)}\right) w_{n}\right) \geqslant-\frac{\epsilon}{2}\|\boldsymbol{w}\|_{A}^{2}-\frac{1}{2 \epsilon} q(\boldsymbol{w}) . \tag{3.59}
\end{equation*}
$$

To prove the inequality above, let's look at

$$
\begin{equation*}
\left\|\epsilon^{1 / 2} \sum_{m=1}^{N_{\text {params }}} \boldsymbol{A}^{(m, l)} \nabla w_{m}+\epsilon^{-1 / 2}\left(\sum_{m=1}^{N_{\text {params }}} \nabla \cdot \boldsymbol{A}^{(m, l)}\right) w_{m}\right\|^{2} \geq 0 \tag{3.60}
\end{equation*}
$$

Setting $x^{(l)}=\sum_{m=1}^{N_{\text {params }}} \boldsymbol{A}^{(m, l)} \nabla w_{m}$ and $y^{(l)}=\left(\sum_{m=1}^{N_{\text {params }}} \nabla \cdot \boldsymbol{A}^{(m, l)}\right) w_{m}$ arrives at

$$
\begin{align*}
& \left\|\epsilon^{1 / 2} x^{(l)}+\epsilon^{-1 / 2} y^{(l)}\right\|^{2} \geq 0  \tag{3.61}\\
\Rightarrow \quad & \left(\epsilon^{1 / 2} x^{(l)}+\epsilon^{-1 / 2} y^{(l)}, \epsilon^{1 / 2} x^{(l)}+\epsilon^{-1 / 2} y^{(l)}\right) \geq 0  \tag{3.62}\\
\Rightarrow \quad & \left\|\epsilon^{1 / 2} x^{(l)}\right\|^{2}+\left\|\epsilon^{-1 / 2} y^{(l)}\right\|^{2}+2\left(x^{(l)}, y^{(l)}\right) \geq 0  \tag{3.63}\\
\Rightarrow \quad & \left(x^{(l)}, y^{(l)}\right) \geq-\frac{\epsilon}{2}\left\|x^{(l)}\right\|^{2}-\frac{1}{2 \epsilon}\left\|y^{(l)}\right\|^{2} . \tag{3.64}
\end{align*}
$$

Summing over $l$

$$
\begin{equation*}
\sum_{l}\left(x^{(l)}, y^{(l)}\right) \geq-\frac{\epsilon}{2} \sum_{l}\left\|x^{(l)}\right\|^{2}-\frac{1}{2 \epsilon} \sum_{l}\left\|y^{(l)}\right\|^{2} \tag{3.65}
\end{equation*}
$$

Then we have the desired result.

Using (3.59) in (3.58) gives

$$
\begin{align*}
b(\boldsymbol{w}, \boldsymbol{w}) & \geqslant\left(1-\frac{\epsilon}{2}\right)\|\boldsymbol{w}\|_{A}^{2}-\frac{1}{2 \epsilon} q(\boldsymbol{w})  \tag{3.66}\\
& \geqslant\left[1-\frac{\epsilon}{2}-\frac{1}{2 \epsilon} C_{p}^{A}\right]\|\boldsymbol{w}\|_{A}^{2}  \tag{3.67}\\
& =C_{1}\|\boldsymbol{w}\|_{A}^{2} \tag{3.68}
\end{align*}
$$

where $C_{1}$ is in the brackets in (3.67); setting $\epsilon=\sqrt{C_{p}^{A}}$ gives

$$
\begin{equation*}
C_{1}=1-\sqrt{C_{p}^{A}} \tag{3.69}
\end{equation*}
$$

Note, we have stability for $C_{p}^{A}<1$.

CAWE Uniqueness Suppose $\boldsymbol{\mu}_{1}$ and $\boldsymbol{\mu}_{2}$ both satisfy Equation (3.43). Let $\boldsymbol{v}=$ $\boldsymbol{\mu}_{1}-\boldsymbol{\mu}_{2} \in \mathcal{V}$. Then according to the bi-linearity of $b(\cdot, \cdot)$, we have

$$
\begin{equation*}
b(\boldsymbol{w}, \boldsymbol{v})=0 \quad \boldsymbol{w} \in \mathcal{V} \tag{3.70}
\end{equation*}
$$

Since $\boldsymbol{v} \in \mathcal{V}$, we may choose $\boldsymbol{w}=\boldsymbol{v}$ to find $b(\boldsymbol{v}, \boldsymbol{v})=0$. By the coercivity of $b(\cdot, \cdot)$, we conclude that $\boldsymbol{v}=\mathbf{0}$, and hence $\boldsymbol{\mu}_{1}=\boldsymbol{\mu}_{2}$. That is, the solution of Equation (3.43) is unique.

### 3.3 Numerical Approximation

We construct a numerical method based on CAWE by approximating the infinite dimensional spaces by their finite dimensional counterparts $\mathcal{V}^{h} \subset \mathcal{V}$ and $\mathcal{S}^{h} \subset \mathcal{S}$. For constructing $\mathcal{V}^{h}$ and $\mathcal{S}^{h} \subset \mathcal{S}$ we use the standard piecewise constant finite element shape functions. The statement of this method is: find $\boldsymbol{\mu}^{h} \in \mathcal{S}^{h}$ such that

$$
\begin{equation*}
b\left(\boldsymbol{w}^{h}, \boldsymbol{\mu}^{h}\right)=l\left(\boldsymbol{w}^{h}\right) \quad \forall \boldsymbol{w}^{h} \in \mathcal{V}^{h} \tag{3.71}
\end{equation*}
$$

Since $\mathcal{S}^{h} \subset \mathcal{S}$ the continuous solution $\mu$ also satisfies Equation (3.71). That is

$$
\begin{equation*}
b\left(\boldsymbol{w}^{h}, \boldsymbol{\mu}\right)=l\left(\boldsymbol{w}^{h}\right) \quad \forall \boldsymbol{w}^{h} \in \mathcal{V}^{h} \tag{3.72}
\end{equation*}
$$

Next we prove that our numerical solution converges at optimal rates to the exact solution under the restrictions of Section 3.2. Denote the Galerkin discretization error by $\boldsymbol{e}=\boldsymbol{\mu}-\boldsymbol{\mu}^{h}$. Then $\boldsymbol{e}$ satisfies

$$
\begin{equation*}
b\left(\boldsymbol{w}^{h}, \boldsymbol{e}\right)=0 \quad \forall \boldsymbol{w}^{h} \in \mathcal{V}^{h} \tag{3.73}
\end{equation*}
$$

We split the error $\boldsymbol{e}=\boldsymbol{\eta}+\boldsymbol{e}^{h}$, where $\boldsymbol{\eta}=\boldsymbol{\mu}-\boldsymbol{\mu}^{i}$ and $\boldsymbol{e}^{h}=\boldsymbol{\mu}^{i}-\boldsymbol{\mu}^{h}$. Here $\boldsymbol{\mu}^{i}$ is the best approximation to $\mu$ in the space $\mathcal{V}^{h}, \boldsymbol{\eta}$ is the interpolation error. Then by linearity of $b(\cdot, \cdot)$ we have:

$$
\begin{equation*}
b\left(\boldsymbol{w}^{h}, \boldsymbol{e}\right)=b\left(\boldsymbol{w}^{h}, \boldsymbol{\eta}+\boldsymbol{e}^{h}\right)=b\left(\boldsymbol{w}^{h}, \boldsymbol{\eta}\right)+b\left(\boldsymbol{w}^{h}, \boldsymbol{e}^{h}\right)=0 \quad \forall \boldsymbol{w}^{h} \in \mathcal{V}^{h} \tag{3.74}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\left|b\left(\boldsymbol{w}^{h}, \boldsymbol{e}^{h}\right)\right|=\left|b\left(\boldsymbol{w}^{h}, \boldsymbol{\eta}\right)\right| \quad \forall \boldsymbol{w}^{h} \in \mathcal{V}^{h} \tag{3.75}
\end{equation*}
$$

Select $\boldsymbol{w}^{h}=\boldsymbol{e}^{h}$ in the equation above to get:

$$
\begin{equation*}
b\left(\boldsymbol{e}^{h}, \boldsymbol{e}^{h}\right)=\left|b\left(\boldsymbol{e}^{h}, \boldsymbol{\eta}\right)\right| \tag{3.76}
\end{equation*}
$$

By continuity of $b(\cdot, \cdot)$ [c.f.[33]], we have

$$
\begin{equation*}
\left|b\left(\boldsymbol{e}^{h}, \boldsymbol{\eta}\right)\right| \leqslant C_{2}\left\|\boldsymbol{e}^{h}\right\|_{A}\|\boldsymbol{\eta}\|_{A} \tag{3.77}
\end{equation*}
$$

By coercivity of $b(\cdot, \cdot)$, we have

$$
\begin{equation*}
b\left(\boldsymbol{e}^{h}, \boldsymbol{e}^{h}\right) \geqslant C_{1}\left\|\boldsymbol{e}^{h}\right\|_{A}^{2} \tag{3.78}
\end{equation*}
$$

Equations (3.76) - (3.77) give

$$
\begin{equation*}
C_{1}\left\|\boldsymbol{e}^{h}\right\|_{A}^{2} \leqslant C_{2}\left\|\boldsymbol{e}^{h}\right\|_{A}\|\boldsymbol{\eta}\|_{A} \tag{3.79}
\end{equation*}
$$

Therefore,

$$
\begin{array}{rll}
\|\boldsymbol{e}\|_{A} & =\left\|\boldsymbol{\eta}+\boldsymbol{e}^{h}\right\|_{A} \\
& \leqslant\|\boldsymbol{\eta}\|_{A}+\left\|\boldsymbol{e}^{h}\right\|_{A} \quad \text { triangle inequality } \\
& \leqslant\left(1+\frac{C_{2}}{C_{1}}\right)\|\boldsymbol{\eta}\|_{A} \quad \text { by }(3.79) \\
& \leqslant\left(1+\frac{C_{2}}{C_{1}}\right) C\|\boldsymbol{\eta}\|_{1} \quad \text { by }(3.57) \\
& \leqslant C_{3} h^{p} . \quad \text { by interpolation estimate } \tag{3.84}
\end{array}
$$

Here $h$ represents the element size and $p$ is the polynomial order of completeness of functions in $\mathcal{V}^{h}$.

We now test the CAWE formulation developed for inverse problems of incompressible plane strain elasticity with synthetically generated data. The synthetic data consists of two cases. The first one is a simple case, i.e. homogeneous medium, where we utilize analytical displacement fields and analytical strains to investigate the ill-posedness of inverse problems of incompressible plane strain elasticity. The second one is a smooth cylindrical inclusion, where we investigate the sensitivity of the data to the inclusion. For the inclusion case, we also consider add Gaussian white noise to the displacement fields to generate $20 \%$ noise in strains and test the performance of the CAWE formulation in the presence of noise. Then we append to the CAWE formulation the total variation diminishing (TVD) regularization that preserves the edges of inclusion and penalizes the variations in the field of interest.

### 3.3.1 Synthetic Data

Homogeneous medium We begin our test with a homogeneous medium with the shear modulus $\mu=1+0.0 i$. The domain of the problem is a square with a mesh of $40 \times 40$ bilinear elements. The mesh size is 0.01 .

We generate analytically two displacement fields in the homogeneous medium,
$\boldsymbol{u}^{(1)}=e^{i k x}[0,1], \boldsymbol{u}^{(2)}=e^{\frac{i k}{\sqrt{2}}(x+y)}[-1,1]$. The wavenumber $k=\sqrt{\frac{\rho \omega^{2}}{\mu}}$ is calculated with the shear modulus $\mu=1+0.0 i$, angular frequency $\omega=30$ and density $\rho=$ 1. To investigate the ill-posedness of this problem, we calculate gradients of the displacement fields $\nabla \boldsymbol{u}$, analytically, to avoid errors introduced in calculating first derivatives of the displacement data. In order to calculate the second derivatives of the displacements, we interpolate $\boldsymbol{\epsilon}$ using piecewise linear finite element basis functions, and then compute their derivatives within each element. In doing so, we introduce numerical error (or noise) into the problem.

We consider the third situation that we discussed in Section 3.1.2, that is to reconstruct the shear modulus in the domain, where two displacement fields are measured and boundary data for the shear modulus is available. According to the analysis in Section 3.1.2, eight real-valued constants are required to obtain the unique solution of the complex-valued shear modulus in inverse problems with two measurements provided, and two real-valued constants are required for each loading to estimate the spacial distribution of the complex-valued pressure uniquely. As shown in Figure 3.2 we prescribe the shear modulus to be equal to its exact value at the four corners of a calibration region, whose width is $1 / 8$ of the total width of the sample. We also prescribe the two pressures at the origin to be equal to zero to get a unique distribution of pressure.

We prefer to impose these conditions weakly through penalty terms. So our weak formulation for reads as: Find $\boldsymbol{\mu}^{h} \in \mathcal{S}^{h}$ such that

$$
\begin{align*}
b\left(\boldsymbol{w}^{h}, \boldsymbol{\mu}^{h}\right) & +\gamma_{1} \operatorname{Re}\left\{\sum_{n=1}^{4} w_{1}^{h}\left(\boldsymbol{x}_{n}\right)\left(\mu^{h}\left(\boldsymbol{x}_{n}\right)-\bar{\mu}^{h}\left(\boldsymbol{x}_{n}\right)\right)\right\} \\
& +\gamma_{2} \operatorname{Re}\left\{w_{2}^{h}\left(\boldsymbol{x}_{0}\right)\left(p^{(1) h}\left(\boldsymbol{x}_{0}\right)-\bar{p}^{(1) h}\left(\boldsymbol{x}_{0}\right)\right)\right\} \\
& +\gamma_{3} \operatorname{Re}\left\{w_{3}^{h}\left(\boldsymbol{x}_{0}\right)\left(p^{(2) h}\left(\boldsymbol{x}_{0}\right)-\bar{p}^{(2) h}\left(\boldsymbol{x}_{0}\right)\right)\right\} \\
& =l\left(\boldsymbol{w}^{h}\right) \quad \forall \boldsymbol{w} \in \mathcal{V}, \tag{3.85}
\end{align*}
$$

where $\gamma_{j}$ is the parameter for weak boundary conditions.
In Figure 3.3 we have shown the reconstructed complex-valued shear modulus using the CAWE formulation. From the reconstruction of the real part and imagi-
nary parts of shear modulus, we note that even though the only noise in the data is in the derivatives of the strains, the results are dominated by artifacts and the maximum error in the reconstruction is $65 \%$. This points to the ill-posedness of this inverse problem [23]. One way to address this problem is to add regularization. In particular, to overcome these artifacts, we can resort to the total variation diminishing (TVD) regularization. However, the case considered here is homogeneous medium and the shear modulus is constant. Extremely large regularization parameter may lead to artificially "perfect" reconstructions of the homogeneous material properties.

The parameter for weak boundary conditions used in Figure 3.3 is $\gamma_{j}=1.0 e 10$. If $\gamma_{j}=1.0 e 5$, that is, we use weaker boundary conditions, the maximum error in the reconstruction goes up to be $75 \%$. We further examine the sensitivity to the value of the boundary conditions. The exact value of the boundary condition is $1.0+0.0 i$. When we use a bigger value of boundary conditions with a relative error $20 \%$, that is $\bar{\mu}=1.2+0.0 i$, the maximum error in the reconstruction increases to be $115 \%$, which is about twice the maximum error in the result with the exact value of the boundary conditions. When we use a smaller value of boundary conditions with a same relative error, that is $\bar{\mu}=0.8+0.0 i$, the maximum error increased to be $145 \%$, which is much larger than the maximum error in the result with a bigger value of boundary conditions.

Inclusion Problem The second example we consider is a two-dimensional plane strain case with a cylindrical inclusion embedded in a homogeneous background. In this example, we investigate the sensitivity of the data to the inclusion. Furthermore, we add Gaussian white noise to the displacement data and test the stability of the CAWE formulation. We also consider the regularized CAWE formulation to improve its performance. Figure 3.4 shows the schematic of the problem setup. The domain of the problem is a $50 \mathrm{~mm} \times 50 \mathrm{~mm}$ square. The inclusion is centered at the center of the domain with a 16 mm diameter. The bottom of the domain is fully constrained and the two lateral sides are traction free. Two time harmonic excitations are located on the top surface and at the top left corner, shown as red

(a)

Figure 3.2: Schematic showing two waves in homogeneous medium. The calibration region is on the left shown as the shaded region. * denotes locations where $\mu$ is prescribed.
arrows in Figure 3.4, to generate two waves propagating vertically and diagonally, respectively. The locations of these two excitations are chosen to make sure that the term $\left(\epsilon_{x x}^{(2)} \times \epsilon_{x y}^{(1)}-\epsilon_{x x}^{(1)} \times \epsilon_{x y}^{(2)}\right)$ is non-zero, which is an assumption that was made to prove the uniqueness of this problem with two measurements in Section 3.1.2. Without this condition, multiple measurement may not reduce the need for boundary data to obtain unique solutions. The frequency of the two excitations is 200 Hz , and the amplitude is 1 mm . The displacement fields generated in these two loadings are numerically measured in the 40 mm square domain, indicated with the dashed line. We use the commercial finite element tool Abaqus to simulate the deformation in the domain.

In the Abaqus model, the background and the inclusion are viscoelastic and nearly incompressible. The density of these two materials is $\rho=1000 \mathrm{~kg} / \mathrm{m}^{3}$. The Poisson's ratio is $\nu=0.495$. In the homogeneous background the instantaneous modulus $\mu_{0}=20 K P a$ and the Prony series parameters $g_{1}=0.14, \tau_{1}=5.0 e-$ $4 s, g_{2}=0.11, \tau_{2}=8.5 e-5 s$. The inclusion has a smooth Cosine profile. Its instantaneous modulus $\mu_{0}=80(\cos (x * \pi / 16))+20 K P a$, starts from the value of the instantaneous modulus of the homogeneous background at the interface and reaches the peak $100 K P a$ at the center of the inclusion. Its Prony series parameters


Figure 3.3: Reconstruction of the shear modulus using CAWE with zero explicit noise displacement fields in homogeneous medium. Left: Real component; Right: Imaginary component.


Figure 3.4: Schematic showing the numerical experimental setup for the inclusion problem. The bottom of the domain is fully constrained and the two lateral sides are traction free. Two time harmonic excitations are located on the top surface (in (a)) and at the top left corner (in (b)) shown as the red arrows, to generate two waves propagating approximately vertically and diagonally, respectively. The frequency of the two excitations is 200 Hz and the amplitude is 1 mm . The calibration region is the top $1 / 8$ of the domain of interest, shown as the shaded region. * denotes locations where $\mu$ is prescribed.

$$
g_{1}=0.14, \tau_{1}=8.5 e-5 s, g_{2}=0.08, \tau_{2}=3.0 e-4 s .
$$

The shear modulus for the background and the inclusion is frequency dependent and can be calculated by [47]

$$
\begin{equation*}
\mu=\mu_{0}\left[1-\sum_{i=1}^{2} g_{i} \frac{1-i \omega \tau_{i}}{1+\omega^{2} \tau_{i}^{2}}\right] . \tag{3.86}
\end{equation*}
$$

Thus the target value of the shear modulus is $\mu_{b g n d}=15.8+i 1.5 \mathrm{KPa}$ in background and $\mu_{\text {incl }}=79+i 4.0 K P a$ at the peak at the center of the inclusion. The maximum contrast of the shear stiffness is about $5: 1$.

The model domain is meshed with $11,213,4$-node bilinear plane strain quadrilateral elements with reduced integration, hourglass control. Implicit, dynamic simulations are performed in Abaqus. Forty cycles of excitation are applied for each loading to get steady state time-dependent response of the domain. The number of increments per each cycle is 20 .

The displacement fields in the last 10 cycles are collected and Fourier transformed to find the displacement fields in frequency domain. The frequency components at the driving frequency $f=200 \mathrm{~Hz}$ are extracted. For each loading, the vector displacement data are scaled such that the maximum magnitude of the vertical displacement component in the domain of interest, as marked in the dashed line in Figure 3.4, is one.

Next, we downsample the displacement data at the driving frequency onto a regular $50 \times 50$ grid using a Matlab function griddata, which fits a surface to the original scatter data and interpolates the surface at the query points [48]. There are three reasons why we downsample and interpolate the displacement data over a regular grid. Firstly, in experiments displacement data are usually measured on a regular grid. Secondly, we want to solve forward problems, which generate the displacement data, and inverse problems on different mesh to avoid an "inverse crime". The last reason is that we are to use a quadratic Least squares filter to smooth the data, which minimizes the least squares error in fitting a quadratic polynomial to frames of noisy data and requires that the mesh be regular. Figure 3.5 shows these two displacement fields. We calculate the strains, $\left(\boldsymbol{a}^{(i)}\right)$, from the gradient of each
displacement component, by solving the following variational problem [49]; Find $\boldsymbol{a}^{(i)}$ such that

$$
\begin{equation*}
\left(\boldsymbol{w}^{h}, \boldsymbol{a}^{(i)}\right)=\left(\boldsymbol{w}^{h}, \nabla u^{(i)}\right), \quad i=1,2 . \tag{3.87}
\end{equation*}
$$

Since the excitations are located on the top of the domain and the waves propagate from top (top-left) to bottom (bottom-right), we assume the boundary data for $\mu$ can be measured somehow in the top part of the domain and the calibration region is the top $1 / 8$ of the specimen (see Figure 3.4). We consider the third situation discussed in Section 3.1.2, that is we resort to two "measured" displacement data to estimate the spatial distribution of the complex valued shear modulus. Therefore, eight real-valued constants are needed to obtain a unique reconstruction of the complex shear modulus, and two real-valued constants are required for each loading to get a unique complex-valued pressure distribution instead of a relative pressure distribution. We choose to impose the boundary data for the shear modulus on the four corners of the calibration region and the boundary data for each pressure field on the top left corner of the calibration region. Once again we apply these data weakly.

We utilize these two displacement fields, their strains and boundary data in the CAWE formulation to recover the spatial distribution of the complex shear modulus. TVD regularization is not used for the clean data. Due to the low ratio of the imaginary part of the shear modulus to the real part of the shear modulus, only the real part of the shear modulus is recovered.

In Figure 3.6 we present the reconstruction of the real part of the shear modulus using CAWE. We observe that the inclusion is well captured, not only in terms of its shape and location, but also in term of its profile. However, the peak value of the smooth inclusion is underestimated by about $10 \%$ and therefore the maximum contrast decreases from $5: 1$ to $4.5: 1$. This phenomenon is also seen in Figure 3.7, in which we plot the variation in the recovered real part of the shear modulus along a horizontal line through the center of the inclusion. There are two primary reasons for the underestimation. The first reason is that in the Abaqus model the center of the inclusion was not on an FEM model and thus the peak value was missed. The


Figure 3.5: Real part of the two vector displacement fields generated in Abaqus. (a) Real part of the horizontal displacement component with excitation on the top edge; (b) Real part of the horizontal displacement component with excitation at the top left corner; (c) Real part of the vertical displacement component with excitation on the top edge; (d) Real part of the vertical displacement component with excitation at the top left corner.
second reason is the displacement data was downsampled onto a uniform, coarse grid using a Matlab function griddata, which determines the connectivity of scattered data based on a Delaunay triangulation and performs a triangle-based linear interpolation to assign values at the " query" points. This process of interpolation further smooths the data.


Figure 3.6: Reconstruction of the real part of the shear modulus using CAWE with two displacement fields. The calibration region is the top part of the domain sharing its top edge but with $1 / 8$ of its height. Boundary data for the shear modulus is imposed weakly at the four corners of the calibration region.

Next, we add $5 \%$ Gaussian white noise to the displacement fields which introduces $20 \%$ noise into the strains and investigate how the CAWE formulation behaves in the presence of noise. For the noisy data we investigate the effect of the total variation diminishing (TVD) regularization and a quadratic least squares filter. Regularization term is simply added to the CAWE formulation. In order to simplify the notation we suppress the superscript $h$. The final CAWE are: Find


Figure 3.7: Variation of the real part of the shear modulus along a horizontal line running through the center of the inclusion with no noise and regularization parameters $\alpha_{j}=0.0$.
$\boldsymbol{\mu} \in \mathcal{S}$ such that

$$
\begin{align*}
b(\boldsymbol{w}, \boldsymbol{\mu}) & +\gamma_{1} \operatorname{Re}\left\{\sum_{n=1}^{4} w_{1}\left(\boldsymbol{x}_{n}\right)\left(\mu\left(\boldsymbol{x}_{n}\right)-\bar{\mu}\left(\boldsymbol{x}_{n}\right)\right)\right\} \\
& +\gamma_{2} \operatorname{Re}\left\{w_{2}\left(\boldsymbol{x}_{0}\right)\left(p^{(1)}\left(\boldsymbol{x}_{0}\right)-\bar{p}^{(1)}\left(\boldsymbol{x}_{0}\right)\right)\right\} \\
& +\gamma_{3} \operatorname{Re}\left\{w_{3}\left(\boldsymbol{x}_{0}\right)\left(p^{(2)}\left(\boldsymbol{x}_{0}\right)-\bar{p}^{(2)}\left(\boldsymbol{x}_{0}\right)\right)\right\} \\
& +\alpha_{1} \operatorname{Re}\left\{\left(\nabla w_{1}^{r}, \frac{\nabla \mu^{r}}{\sqrt{\left|\nabla \mu^{r}\right|^{2}+\beta^{2}}}\right)\right\} \\
& +\alpha_{2} \operatorname{Re}\left\{\left(\nabla w_{1}^{i}, \frac{\nabla \mu^{i}}{\sqrt{\left|\nabla \mu^{i}\right|^{2}+\beta^{2}}}\right)\right\} \\
& +\alpha_{3} \operatorname{Re}\left\{\left(\nabla w_{2}^{r}, \frac{\nabla p^{(1) r}}{\sqrt{\left|\nabla p^{(1) r}\right|^{2}+\beta^{2}}}\right)\right\} \\
& +\alpha_{4} \operatorname{Re}\left\{\left(\nabla w_{2}^{i}, \frac{\nabla p^{(1) i}}{\sqrt{\left|\nabla p^{(1) i}\right|^{2}+\beta^{2}}}\right)\right\} \\
& +\alpha_{5} \operatorname{Re}\left\{\left(\nabla w_{3}^{r}, \frac{\nabla p^{(2) r}}{\sqrt{\left|\nabla p^{(2) r}\right|^{2}+\beta^{2}}}\right)\right\} \\
& +\alpha_{6} \operatorname{Re}\left\{\left(\nabla w_{3}^{i}, \frac{\nabla p^{(2) i}}{\sqrt{\left|\nabla p^{(2) i}\right|^{2}+\beta^{2}}}\right)\right\} \\
& =l(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \mathcal{V}, \tag{3.88}
\end{align*}
$$

where $\alpha_{j}$ is the regularization parameter and $\beta$ is a parameter selected to ensure that the regularization term is continuous when $\nabla \mu^{r}=\mathbf{0}, \nabla \mu^{i}=\mathbf{0}, \nabla p^{(1) r}=\mathbf{0}$, $\nabla p^{(1) i}=\mathbf{0}, \nabla p^{(2) r}=\mathbf{0}$ or $\nabla p^{(2) i}=\mathbf{0} . \gamma_{j}$ are penalty parameters for imposing the weak boundary conditions.

The comparison of the results for the real part of the shear modulus reconstructions are presented in Figure 3.8. Figure 3.8(a) shows the reconstruction with noisy data and no TVD regularization applied. The location of the inclusion is well detected. However, the shape of the inclusion is not well captured. The smooth profile of the inclusion is interrupted by sharp oscillations and significant artifacts are also present in the homogeneous background. The peak value of the inclusion is around 60-70 KPa regardless the overshoots caused by noise, which is very close to the recovered peak value with zero noise data shown in Figure 3.6. Then we append to the CAWE formulation the TVD regularization to improve the reconstruction. The corresponding result is shown in Figure 3.8(b). We observe that with the TVD regularization the shape of the inclusion is recovered better, the oscillations in the inclusion and artifacts in the background are diminished. We also note that the peak value of the inclusion is decreased due to the use of the regularization. We can expect that the oscillations and artifacts in the reconstruction may be tempered very well but at the expense of loosing contrast by increasing the regularization parameter. Thereafter, we smooth the noisy displacement data by using a quadratic least squares filter with window size of $5 \times 5$ and perform the inversion algorithm. Figure 3.8(c) and Figure 3.8(d) show the reconstructions of the real part of the shear modulus from the smoothed data with/without the TVD regularization, respectively. We observe that in the reconstruction with smooth data (Figure 3.8(c)) overshoots are smoothed. The smooth profile of the inclusion is recovered successfully. The value of the real part of the shear modulus in the inclusion is close to the result with noisy data, regardless of the overshoots caused by noise. Then we consider using the TVD regularization. We adopt the same TVD regularization parameters that we used for noisy data and present the result in Figure 3.8(d). We find that the shape of the inclusion is reconstructed very well. The artifacts are removed except the small region below the inclusion. However the contrast in the real part of the
shear modulus between the inclusion and the background is overly underestimated. The maximum contrast in the reconstruction is $2: 1$, which is lower than the exact value of $5: 1$. The real part of the shear modulus in the inclusion is flattened. These observations are reaffirmed by the plot in Figure 3.9, where the recovered real part of the shear modulus along a horizontal line across the center of the inclusion are plotted.

### 3.4 Chapter Summary

We considered the inverse problem of time-harmonic incompressible isotropic plane strain elasticity. We examined the uniqueness of this problem and determined the boundary data required to generate unique solutions for the complex valued shear modulus. We concluded that the need for boundary data depends on the number of available measurements. When a single measurement is available, the problem is elliptic in most cases but could be hyperbolic in some particular case. The data required to determine the solution uniquely is difficult to gather in practice. When two measurements are available, the two equations for the shear modulus can be manipulated to yield two hyperbolic partial differential equations. Then the need for boundary data is significantly reduced. Only eight real-valued constants are required to estimate the unique solution of the complex valued shear modulus, and two real-valued constants for each pressure field are needed to find the unique solution for the complex valued pressure.

In order to solve this problem, we developed the CAWE formulation for the case of multiple measurements and parameters. We proved that the CAWE formulation yield a method that is stable and convergent under some restrictions on measured data. We appended the total variation diminishing (TVD) regularization to the CAWE formulation, in order to penalize variations in the field of interest while preserving the sharpness of jumps at the interface of two different materials. We also developed a weak formulation to apply the $8+2$ constants for the two-measurement case.

We implemented a straightforward finite element discretization of the regularized CAWE formulation. Thereafter we evaluated the performance of this algorithm
through synthetically generated, noisy displacement data. We found the regularized CAWE formulation accommodated noise gracefully and recovered the shape, location and value of the shear modulus successfully.


Figure 3.8: Reconstructions of the real part of the shear modulus using CAWE with noisy displacement fields. (a) With no smoothing of the noisy data ( $s=0$ ) and regularization parameters $\alpha_{j}=0(j=1, \cdots, 6) . s$ is the window size of the quadratic LS filter; (b) With no smoothing $(s=0)$ and regularization parameters $\alpha_{j}=20$. (c) With smoothing $(s=5)$ and regularization parameters $\alpha_{j}=0$; (d) With smoothing $(s=5)$ and regularization parameters $\alpha_{j}=20$.

(a)

Figure 3.9: Variation of the real part of the shear modulus along a horizontal line running through the center of the inclusion for four cases: (1) With no smoothing $(s=0)$ and regularization parameters $\alpha_{j}=0(j=1, \cdots, 6) . s$ is the window size of the quadratic LS filter; (2) With no smoothing ( $s=0$ ) and regularization parameters $\alpha_{j}=20$; (3) With smoothing ( $s=5$ ) and regularization parameters $\alpha_{j}=0$; (4) With smoothing ( $s=5$ ) and regularization parameters $\alpha_{j}=20$.

## CHAPTER 4 Three-Dimensional Time-Harmonic Viscoelasticity Problem

In this chapter we extend the CAWE formulation developed for simplified mathematical models that include the scalar Helmholtz equation, the anti-plane shear, the plane stress and the plane strain states to three-dimensional time-harmonic viscoelastic problems. We also examine and describe the uniqueness of the 3D inverse problem with one and two measurements. The motivation for this problem lies in the following two considerations. Firstly, these simplified mathematical models are only approximations of the actual state, which is usually three dimensional. In this regard, studies [30] have shown that although these approximations simplify and speed up calculations, they also introduce inaccuracy in the recovered shear modulus distribution. Secondly, three dimensional data allows us to recover and track the changes of the shear modulus in the elevation direction. With this background, we carry out studies of the three-dimensional time-harmonic viscoelastic inverse problem and develop a CAWE formulation to solve it directly.

At the beginning of this chapter, we write the 3D time-harmonic viscoelastic equations. We analyze these equations as a problem for the shear modulus given the displacement field, and determine the boundary data and/or the interior displacement data that would be required to generate a unique solution for the complexvalued shear modulus. As in the plane strain case considered in Chapter 3, we consider three situations: (1) A single measured displacement field with boundary data available for shear modulus $\mu$ and pressure $p$; (2) A single measured displacement field with boundary data available only for the shear modulus $\mu$; and (3) two measured displacement fields with limited data available only for the shear modulus $\mu$. We note that the three-dimensional case is very similar to the two-dimensional case considered in Chapter 3. Hence we rely quite heavily on the development contained within that chapter. We begin by considering the case with one measured displacement field and conclude that the resulting system of equations (for $\mu \& p$, or for $\mu$ alone) could be hyperbolic, parabolic or elliptic depending on a condition on
the strain fields. Thereafter we consider the case with two measured displacement fields. Here we conclude that the equations can be manipulated in a way such that the resulting system of equations is purely hyperbolic. We make extensive use of this property to establish the uniqueness of this problem. We conclude that the solution for $\mu$ is unique given up to eight pieces of real data on $\mu$.

We also develop the CAWE formulation for the 3D time-harmonic viscoelasticity inverse problem. We find that the general CAWE formulation developed in Chapter 3 and its properties came forward to the inverse 3D problem. Finally we test the CAWE formulation on magnetic resonance (MR) data.

The layout of the remainder of this chapter is as follows. In Section 4.1 we present the three-dimensional time-harmonic viscoelasticity inverse problem and analyze its uniqueness. In Section 4.2 we describe the CAWE formulation in 3D and analyze its stability and convergence under certain conditions. Thereafter in Section 4.3 we consider the Galerkin discretization and test the CAWE formulation using experimentally measured data. We end with conclusions in Section 4.4.

### 4.1 Strong Form

### 4.1.1 Problem Formulation

From the momentum equation (2.1) and the constitutive equation (2.3), we obtain the governing equation for three-dimensional time-harmonic viscoelasticity problem

$$
\begin{equation*}
-\nabla p+\nabla \cdot(2 \mu \boldsymbol{\epsilon})+\rho \omega^{2} \boldsymbol{u}=\mathbf{0} \tag{4.1}
\end{equation*}
$$

where $\boldsymbol{u}(\boldsymbol{x})=\left[u_{x}(x, y, z), u_{y}(x, y, z), u_{z}(x, y, z)\right]^{T}$ is the vector displacement field measured in a three-dimensional volume. $\boldsymbol{\epsilon}=\frac{1}{2}\left(\nabla \boldsymbol{u}+\nabla \boldsymbol{u}^{T}\right)$ is the second-order strain tensor. Here we consider the inverse problem, that is, we wish to find the complex-valued shear modulus, $\mu(\boldsymbol{x})$, and the hydrostatic stress, $p(\boldsymbol{x})$, fields that satisfy the above equation with the displacement field $\boldsymbol{u}(\boldsymbol{x})$ and strain $\boldsymbol{\epsilon}$ given.

In what follows, we shall examine the uniqueness of the three-dimensional time-harmonic viscoelasticity inverse problem. In this regard we classify the type
of PDE for $\mu$ and then determine the data required to find a unique solution. We will consider cases, where one or more measured displacements are known, and the spatial distribution of the complex-valued shear modulus is sought.

### 4.1.2 Uniqueness Results

We examine the uniqueness of the problem under three conditions: (1) A single measured displacement field with boundary data available for $\mu$ and $p$; (2) A single measured displacement field with boundary data available only for $\mu$; and (3) two measured displacement fields with boundary data available only for $\mu$.

## Single displacement field with boundary data available for $\mu$ and $p$ We

 note that Equation (4.1) gives a system of three equations and is for the two unknowns $\mu$ and $p$. Thus we take two of these three equations to characterize this system:$$
\begin{align*}
-p_{, x}+2 \mu_{, x} \epsilon_{x x}+2 \mu_{, y} \epsilon_{x y}+2 \mu_{, z} \epsilon_{x z}+2 \mu \epsilon_{x j, j}+\rho \omega^{2} u_{x} & =0  \tag{4.2}\\
-p_{, y}+2 \mu_{, x} \epsilon_{y x}+2 \mu_{, y} \epsilon_{y y}+2 \mu_{, z} \epsilon_{y z}+2 \mu \epsilon_{y j, j}+\rho \omega^{2} u_{y} & =0 . \tag{4.3}
\end{align*}
$$

We write the corresponding real system of equations obtained for the realvalued vector field $\boldsymbol{\mu}=\left[\mu^{r}, \mu^{i}, p^{r}, p^{i}\right]^{T}$ :

$$
\begin{equation*}
\boldsymbol{A} \boldsymbol{\mu}_{, x}+\boldsymbol{B} \boldsymbol{\mu}_{, y}+\boldsymbol{C} \boldsymbol{\mu}_{, \boldsymbol{z}}+\boldsymbol{D} \boldsymbol{\mu}+\rho \omega^{2} \boldsymbol{u}=\mathbf{0} \tag{4.4}
\end{equation*}
$$

Here

$$
\begin{aligned}
& \boldsymbol{A}=\left[\begin{array}{cccc}
2 \epsilon_{x x}^{r} & -2 \epsilon_{x x}^{i} & -1 & 0 \\
2 \epsilon_{x x}^{i} & 2 \epsilon_{x x}^{r} & 0 & -1 \\
2 \epsilon_{x y}^{r} & -2 \epsilon_{x y}^{i} & 0 & 0 \\
2 \epsilon_{x y}^{i} & 2 \epsilon_{x y}^{r} & 0 & 0
\end{array}\right] \\
& \boldsymbol{B}=\left[\begin{array}{cccc}
2 \epsilon_{x y}^{r} & -2 \epsilon_{x y}^{i} & 0 & 0 \\
2 \epsilon_{x y}^{i} & 2 \epsilon_{x y}^{r} & 0 & 0 \\
2 \epsilon_{y y}^{r} & -2 \epsilon_{y y}^{i} & -1 & 0 \\
2 \epsilon_{y y}^{i} & 2 \epsilon_{y y}^{r} & 0 & -1
\end{array}\right] \\
& \boldsymbol{C}=\left[\begin{array}{cccc}
2 \epsilon_{x z}^{r} & -2 \epsilon_{x z}^{i} & 0 & 0 \\
2 \epsilon_{x z}^{i} & 2 \epsilon_{x z}^{r} & 0 & 0 \\
2 \epsilon_{y z}^{r} & -2 \epsilon_{y z}^{i} & 0 & 0 \\
2 \epsilon_{y z}^{i} & 2 \epsilon_{y z}^{r} & 0 & 0
\end{array}\right] \\
& \boldsymbol{D}=\left[\begin{array}{llll}
2 \epsilon_{x j, j}^{r} & -2 \epsilon_{x j, j}^{i} & 0 & 0 \\
2 \epsilon_{j j, j}^{i} & 2 \epsilon_{x j, j}^{r} & 0 & 0 \\
2 \epsilon_{y j, j}^{r} & -2 \epsilon_{y j, j}^{i} & 0 & 0 \\
2 \epsilon_{y j, j}^{i} & 2 \epsilon_{y j, j}^{r} & 0 & 0
\end{array}\right] \\
& \boldsymbol{u}=\left[\begin{array}{lll}
u_{x}^{r} \\
u_{x}^{i} \\
u_{y}^{r} \\
u_{y}^{i}
\end{array}\right] .
\end{aligned}
$$

The classification for the system of PDEs above and hence the required boundary data is determined by the characteristic equation $\operatorname{det}\left(\boldsymbol{A}+\tau_{1} \boldsymbol{B}+\tau_{2} \boldsymbol{C}\right)=0$. For arbitrarily prescribed real values of $\tau_{1}$, if this characteristic equation does not possess real roots (or called eigenvalues) $\tau_{2}$, then the problem is elliptic; if this characteristic equation possesses four real distinct solutions $\tau_{2}$, or the solutions to this equation are real and the system is not defective, then this problem is hyperbolic; if the solutions to this equation are real, but the system is defective, then this system is parabolic. A system is said to be non defective if the algebraic and the geometric multiplicities
of each eigenvalue are identical.

$$
\begin{aligned}
& \operatorname{det}\left(\boldsymbol{A}+\tau_{1} \boldsymbol{B}+\tau_{2} \boldsymbol{C}\right) \\
& =\operatorname{det}\left[\begin{array}{cccc}
2\left(\epsilon_{x x}^{r}+\tau_{1} \epsilon_{x y}^{r}+\tau_{2} \epsilon_{x z}^{r}\right) & -2\left(\epsilon_{x x}^{i}+\tau_{1} \epsilon_{x y}^{i}+\tau_{2} \epsilon_{x z}^{i}\right) & -1 & 0 \\
2\left(\epsilon_{x x}^{i}+\tau_{1} \epsilon_{x y}^{i}+\tau_{2} \epsilon_{x z}^{i}\right) & 2\left(\epsilon_{x x}^{r}+\tau_{1} \epsilon_{x y}^{r}+\tau_{2} \epsilon_{x z}^{r}\right) & 0 & -1 \\
2\left(\epsilon_{x y}^{r}+\tau_{1} \epsilon_{y y}^{r}+\tau_{2} \epsilon_{y z}^{r}\right) & -2\left(\epsilon_{x y}^{i}+\tau_{1} \epsilon_{y y}^{i}+\tau_{2} \epsilon_{y z}^{i}\right) & -\tau_{1} & 0 \\
2\left(\epsilon_{x y}^{i}+\tau_{1} \epsilon_{y y}^{i}+\tau_{2} \epsilon_{y z}^{i}\right) & 2\left(\epsilon_{x y}^{r}+\tau_{1} \epsilon_{y y}^{r}+\tau_{2} \epsilon_{y z}^{r}\right) & 0 & -\tau_{1}
\end{array}\right] \\
& =4\left[\left(\epsilon_{x y}^{r} \tau_{1}^{2}+\left(\epsilon_{x x}^{r}-\epsilon_{y y}^{r}\right) \tau_{1}-\epsilon_{x y}^{r}\right)^{2}+\left(\epsilon_{x z}^{r} \tau_{1}-\epsilon_{y z}^{r}\right) \tau_{2}\right]^{2} \\
& \quad+4\left[\left(\epsilon_{x y}^{i} \tau_{1}^{2}+\left(\epsilon_{x x}^{i}-\epsilon_{y y}^{i}\right) \tau_{1}-\epsilon_{x y}^{i}\right)^{2}+\left(\epsilon_{x z}^{i} \tau_{1}-\epsilon_{y z}^{i}\right) \tau_{2}\right]^{2} .
\end{aligned}
$$

Setting $\operatorname{det}\left(\boldsymbol{A}+\tau_{1} \boldsymbol{B}+\tau_{2} \boldsymbol{C}\right)=0$ and assuming $\tau_{1} \& \tau_{2}$ are real, the above implies

$$
\begin{align*}
& \left(\left(\epsilon_{x y}^{r} \tau_{1}^{2}+\left(\epsilon_{x x}^{r}-\epsilon_{y y}^{r}\right) \tau_{1}-\epsilon_{x y}^{r}\right)^{2}+\left(\epsilon_{x z}^{r} \tau_{1}-\epsilon_{y z}^{r}\right) \tau_{2}\right)^{2}=0, \quad \text { and }  \tag{4.5}\\
& \left(\left(\epsilon_{x y}^{i} \tau_{1}^{2}+\left(\epsilon_{x x}^{i}-\epsilon_{y y}^{i}\right) \tau_{1}-\epsilon_{x y}^{i}\right)^{2}+\left(\epsilon_{x z}^{i} \tau_{1}-\epsilon_{y z}^{i}\right) \tau_{2}\right)^{2}=0 \tag{4.6}
\end{align*}
$$

For each of Equation (4.5) and Equation (4.5), we observe that at most one solution for $\tau_{2}$ exist. Therefore, for this system to be hyperbolic we have to find a real $\tau_{2}$ that satisfies Equation (4.5) \& Equation (4.6) simultaneously for arbitrary values of $\tau_{1}$, and determine the system is not defective. If the real $\tau_{2}$ exists but the system is defective, this system is parabolic. If no real $\tau_{2}$ exists and satisfies Equation (4.5) \& Equation (4.6) simultaneously for arbitrary real $\tau_{1}$, this system is elliptic. There are three possibilities for $\tau_{1}$.

1. When

$$
\begin{equation*}
\tau_{1}=\epsilon_{y z}^{r} / \epsilon_{x z}^{r} \tag{4.7}
\end{equation*}
$$

then for Equation (4.5) to hold we must have

$$
\begin{equation*}
\tau_{1}=\frac{\left(\epsilon_{y y}^{r}-\epsilon_{x x}^{r}\right) \pm\left(\left(\epsilon_{y y}^{r}-\epsilon_{x x}^{r}\right)^{2}+4 \epsilon_{x y}^{r 2}\right)^{1 / 2}}{2 \epsilon_{x y}^{r}} . \tag{4.8}
\end{equation*}
$$

Now unless the strains are such that the RHS of Equation (4.7) and Equation
(4.8) are the same, we have found a value of $\tau_{1}$ for which no real value of $\tau_{2}$ will ensure that the $\operatorname{det}\left(\boldsymbol{A}+\tau_{1} \boldsymbol{B}+\tau_{2} \boldsymbol{C}\right)=0$.
2. When

$$
\begin{equation*}
\tau_{1}=\epsilon_{y z}^{i} / \epsilon_{x z}^{i} \tag{4.9}
\end{equation*}
$$

we obtain a similar constraint on the strains from Equation (4.6).
3. When $\left(\epsilon_{x z}^{r} \tau_{1}-\epsilon_{y z}^{r}\right) \neq 0$ and $\left(\epsilon_{x z}^{i} \tau_{1}-\epsilon_{y z}^{i}\right) \neq 0$, from Equation (4.5) we have

$$
\begin{equation*}
\tau_{2}=-\frac{\left(\epsilon_{x y}^{r} \tau_{1}^{2}+\left(\epsilon_{x x}^{r}-\epsilon_{y y}^{r}\right) \tau_{1}-\epsilon_{x y}^{r}\right)^{2}}{\left(\epsilon_{x z}^{r} \tau_{1}-\epsilon_{y z}^{r}\right)} \tag{4.10}
\end{equation*}
$$

and from Equation (4.6) we have

$$
\begin{equation*}
\tau_{2}=-\frac{\left(\epsilon_{x y}^{i} \tau_{1}^{2}+\left(\epsilon_{x x}^{i}-\epsilon_{y y}^{i}\right) \tau_{1}-\epsilon_{x y}^{i}\right)^{2}}{\left(\epsilon_{x z}^{i} \tau_{1}-\epsilon_{y z}^{i}\right)} \tag{4.11}
\end{equation*}
$$

Once again for this to be true we require a special condition on the strain field.

Therefore, we can conclude that the system is elliptic in most cases, but can be hyperbolic or parabolic if the strain field satisfies certain restrictive boundary conditions.

Single displacement field with boundary data available only for $\mu$ Since the boundary data for $p$ is not available in this situation, we take curl of Equation (4.1) to eliminate the pressure term and arrive at

$$
\begin{equation*}
\nabla \times(\nabla \cdot(2 \mu \boldsymbol{\epsilon}))=\mathbf{0} \tag{4.12}
\end{equation*}
$$

expressed in indicial notation as,

$$
\begin{equation*}
\boldsymbol{e}_{k} \varepsilon_{k l m} \partial_{l}\left(2 \mu_{, j} \epsilon_{m j}+2 \mu \epsilon_{m j, j}\right)=0 \tag{4.13}
\end{equation*}
$$

where $\boldsymbol{e}_{k}$ are the coordinate vector fields and $\varepsilon_{k l m}$ is the Levi-Civita symbol, that is

$$
\varepsilon_{k l m}= \begin{cases}+1 & \text { if }(k, l, m) \text { is }(1,2,3),(3,1,2) \text { or }(2,3,1)  \tag{4.14}\\ -1 & \text { if }(k, l, m) \text { is }(1,3,2),(3,2,1) \text { or }(2,1,3) \\ 0 & \text { if } k=l \text { or } l=m \text { or } m=k\end{cases}
$$

Equation (4.13) gives us a system of three equations for a single unknown, that is shear modulus $\mu$ :

$$
\begin{array}{r}
\epsilon_{y z}\left(\mu_{, y y}-\mu_{, z z}\right)+\left(\epsilon_{x z} \mu_{, x y}+\left(\epsilon_{z z}-\epsilon_{y y}\right) \mu_{, y z}+\left(-\epsilon_{x y}\right) \mu_{, x z}\right) \\
+\left(\left(\epsilon_{x z, y}-\epsilon_{x y, z}\right) \mu_{, x}+\left(\epsilon_{y z, y}+\epsilon_{z j, j}-\epsilon_{y y, z}\right) \mu_{, y}+\left(\epsilon_{z z, y}-\epsilon_{z y, z}-\epsilon_{y j, j}\right) \mu_{, z}\right) \\
+\left(\epsilon_{z j, j y}-\epsilon_{y j, j z}\right) \mu+d_{1}=0 \\
\epsilon_{x z}\left(\mu_{, z z}-\mu_{, x x}\right)+\left(\left(-\epsilon_{y z}\right) \mu_{, x y}+\epsilon_{x y} \mu_{, y z}+\left(\epsilon_{x x}-\epsilon_{z z}\right) \mu_{, x z}\right) \\
\left.+\left(\left(\epsilon_{x x, z}-\epsilon_{x z, x}-\epsilon_{z j, j}\right) \mu_{, x}+\left(\epsilon_{x y, z}-\epsilon_{y z, x}\right)\right) \mu_{, y}+\left(\epsilon_{x z, z}+\epsilon_{x j, j}-\epsilon_{z z, x}\right) \mu_{, z}\right) \\
+\left(\epsilon_{x j, j z}-\epsilon_{z j, j x}\right) \mu+d_{2}=0 \\
\epsilon_{x y}\left(\mu_{, x x}-\mu_{, y y}\right)+\left(\left(\epsilon_{y y}-\epsilon_{x x}\right) \mu_{, x y}-\epsilon_{x z} \mu_{, y z}+\epsilon_{y z} \mu_{, x z}\right) \\
+\left(\left(\epsilon_{y x, x}+\epsilon_{y j, j}-\epsilon_{x x, y}\right) \mu_{, x}+\left(\epsilon_{y y, x}-\epsilon_{x y, y}-\epsilon_{x j, j}\right) \mu_{, y}+\left(\epsilon_{y z, x}-\epsilon_{x z, y}\right) \mu_{, z}\right) \\
+\left(\epsilon_{y j, j x}-\epsilon_{x j, j y}\right) \mu+d_{3}=0 . \tag{4.17}
\end{array}
$$

Two displacement fields with limited data available only for $\mu$ Here we assume that two displacement fields are given and thus we rewrite Equations (4.15)(4.17) for each displacement field

$$
\begin{align*}
& a_{1}\left(\mu_{, y y}-\mu_{, z z}\right)+\left(b_{1} \mu_{, x y}+c_{1} \mu_{, y z}+d_{1} \mu_{, x z}\right) \\
& \quad+\left(e_{1} \mu_{, x}+f_{1} \mu_{, y}+g_{1} \mu_{, z}\right)+h_{1} \mu+k_{1}=0  \tag{4.18}\\
& a_{2}\left(\mu_{, z z}-\mu_{, x x}\right)+\left(b_{2} \mu_{, x y}+c_{2} \mu_{, y z}+d_{2} \mu_{, x z}\right) \\
& \quad+\left(e_{2} \mu_{, x}+f_{2} \mu_{, y}+g_{2} \mu_{, z}\right)+h_{2} \mu+k_{2}=0  \tag{4.19}\\
& a_{3}\left(\mu_{, x x}-\mu_{, y y}\right)+\left(b_{3} \mu_{, x y}+c_{3} \mu_{, y z}+d_{3} \mu_{, x z}\right) \\
& \quad+\left(e_{3} \mu_{, x}+f_{3} \mu_{, y}+g_{3} \mu_{, z}\right)+h_{3} \mu+k_{3}=0 \tag{4.20}
\end{align*}
$$

and

$$
\begin{align*}
& a_{4}\left(\mu_{, y y}-\mu_{, z z}\right)+\left(b_{4} \mu_{, x y}+c_{4} \mu_{, y z}+d_{4} \mu_{, x z}\right) \\
& \quad+\left(e_{4} \mu_{, x}+f_{4} \mu_{, y}+g_{4} \mu_{, z}\right)+h_{4} \mu+k_{4}=0  \tag{4.21}\\
& a_{5}\left(\mu_{, z z}-\mu_{, x x}\right)+\left(b_{5} \mu_{, x y}+c_{5} \mu_{, y z}+d_{5} \mu_{, x z}\right) \\
& \quad+\left(e_{5} \mu_{, x}+f_{5} \mu_{, y}+g_{5} \mu_{, z}\right)+h_{5} \mu+k_{5}=0  \tag{4.22}\\
& a_{6}\left(\mu_{, x x}-\mu_{, y y}\right)+\left(b_{6} \mu_{, x y}+c_{6} \mu_{, y z}+d_{6} \mu_{, x z}\right) \\
& \quad+\left(e_{6} \mu_{, x}+f_{6} \mu_{, y}+g_{6} \mu_{, z}\right)+h_{6} \mu+k_{6}=0 . \tag{4.23}
\end{align*}
$$

Equations (4.18)-(4.20) are from the first displacement field and Equations (4.21)(4.23) are from the second displacement field. The coefficients $a_{j}, \cdots k_{j}$ are functions of the displacement components and their derivatives. In the following calculations we shall keep introducing new coefficients $a_{j}, \cdots k_{j}$ to keep the notation simple.

Totally, given two displacement fields, we have six equations for the threedimensional time-harmonic viscoelasticity problem. We observe that four terms that contain $z$-derivatives of $\mu$ appear in these equations, that is $\mu_{, z}, \mu_{, x z}, \mu_{, y z}, \mu_{, z z}$. Hence, we can expect to get two independent equations only involving $x$ - and $y$ derivatives, These partial differential equations, when restricted to the $x y$ plane have the same form of the PDEs that we have considered for the two-dimensional plane strain problem in Section 3.1.2. According to the analysis in that section, we know that eight real-valued constants are required to guarantee a unique solution to this problem. For the 3D inverse problem, once we assume that we know these eight constants, we can solve for $\mu$ in $x y$ plane. Thereafter we use the remaining four equations involving $z$-derivatives of $\mu$ to calculate $\mu_{, z}$. With this we obtain sufficient Cauchy data in any 2D plane, which is parallel to $x z$ or $y z$ plane. In this way we can determine the shear modulus in the entire 3D volume. In what follows, we carry out these steps in detail.

To get at the two equations involving only $x$ - and $y$-derivatives, we start from Equations (4.20) and (4.23). We express $\mu_{, x z}, \mu_{, y z}$ in terms of $x-, y$-, $z$ derivatives, eliminate the term $\mu_{, z}$ and finally get two equations only involving $x$ -
and $y$-derivatives. To this end, first we take $a_{4} \times(4.18)-a_{1} \times(4.21)$ to get

$$
\begin{equation*}
\left(b_{7} \mu_{, x y}+c_{7} \mu_{, y z}+d_{7} \mu_{, x z}\right)+\left(e_{7} \mu_{, x}+f_{7} \mu_{, y}+g_{7} \mu_{, z}\right)+h_{7} \mu+k_{7}=0 \tag{4.24}
\end{equation*}
$$

Similarly, we calculate $a_{5} \times(4.19)-a_{2} \times(4.22)$ to obtain

$$
\begin{equation*}
\left(b_{8} \mu_{, x y}+c_{8} \mu_{, y z}+d_{8} \mu_{, x z}\right)+\left(e_{8} \mu_{, x}+f_{8} \mu_{, y}+g_{8} \mu_{, z}\right)+h_{8} \mu+k_{8}=0 \tag{4.25}
\end{equation*}
$$

From Equations (4.24) and (4.25), we solve for $\mu_{, x z}$ and $\mu_{, y z}$

$$
\begin{align*}
\mu_{, x z} & =F_{1}\left(\mu_{, x y}, \mu_{, x}, \mu_{, y}, \mu_{, z}, \mu\right)  \tag{4.26}\\
\mu_{, y z} & =F_{2}\left(\mu_{, x y}, \mu_{, x}, \mu_{, y}, \mu_{, z}, \mu\right) \tag{4.27}
\end{align*}
$$

We substitute Equations (4.26) and (4.27) into Equations (4.20) and (4.23) to obtain

$$
\begin{align*}
a_{9}\left(\mu_{, x x}-\mu_{, y y}\right)+b_{9} \mu_{, x y}+\left(e_{9} \mu_{, x}+f_{9} \mu_{, y}+g_{9} \mu_{, z}\right) & \\
+h_{9} \mu+k_{9} & =0  \tag{4.28}\\
a_{10}\left(\mu_{, x x}-\mu_{, y y}\right)+b_{10} \mu_{, x y}+\left(e_{10} \mu_{, x}+f_{10} \mu_{, y}+g_{10} \mu_{, z}\right) & \\
+h_{10} \mu+k_{10} & =0 . \tag{4.29}
\end{align*}
$$

Next we shall eliminate $\mu_{, z}$. To do this, we first take $a_{2} \times(4.18)+a_{1} \times(4.19)$ to eliminate $\mu_{, z z}$, replace $\mu_{, x z}, \mu_{, y z}$ by terms involving $\mu_{, x y}, \mu_{, x}, \mu_{, y}, \mu_{, z}, \mu$ (Equations (4.26) and (4.27)) and then arrive at

$$
\begin{align*}
& a_{11}\left(\mu_{, x x}-\mu_{, y y}\right)+b_{11} \mu_{, x y}+\left(e_{11} \mu_{, x}+f_{11} \mu_{, y}+g_{11} \mu_{, z}\right) \\
&+h_{11} \mu+k_{11}=0 \tag{4.30}
\end{align*}
$$

We solve the above equation for $\mu_{, z}$

$$
\begin{equation*}
\mu_{, z}=F_{3}\left(\mu_{, x x}, \mu_{, y y}, \mu_{, x y}, \mu_{, x}, \mu_{, y}, \mu\right) \tag{4.31}
\end{equation*}
$$

Finally, we substitute Equation (4.31) into Equations (4.28) and (4.29) to obtain

$$
\begin{align*}
& a_{12}\left(\mu_{, x x}-\mu_{, y y}\right)+b_{12} \mu_{, x y}+e_{12} \mu_{, x}+f_{12} \mu_{, y}+h_{12} \mu+k_{12}=0  \tag{4.32}\\
& a_{13}\left(\mu_{, x x}-\mu_{, y y}\right)+b_{13} \mu_{, x y}+e_{13} \mu_{, x}+f_{13} \mu_{, y}+h_{13} \mu+k_{13}=0 . \tag{4.33}
\end{align*}
$$

We note that the equation system (4.32)-(4.33) above, when restricted to the $x y$ plane, coincides with the equation system (3.14)-(3.15) for the two-dimensional plane strain problem with two measured displacement fields.

We know from the analysis in Section 3.1.2 that with up to eight real-valued data for $\mu$ we are able to solve the 2D plane strain problem for a unique solution of $\mu$. Hence, given up to eight real-valued data for $\mu$ in the $x y$ plane, we are able to solve the equation system (4.32)-(4.33) and obtain a unique solution for $\mu$ in $x y$ plane, that is

$$
\begin{equation*}
\mu(x, y, 0)=\bar{\mu}(x, y) \tag{4.34}
\end{equation*}
$$

We substitute the above equation into Equation (4.31) to solve for $\mu_{, z}$

$$
\begin{equation*}
\mu_{, z}(x, y, 0)=F_{3}\left(\bar{\mu}_{, x x}, \bar{\mu}_{, y y}, \bar{\mu}_{, x y}, \bar{\mu}_{, x}, \bar{\mu}_{, y}, \bar{\mu}\right) . \tag{4.35}
\end{equation*}
$$

Now we have known $\mu$ in $x y$ plane and its $z$ derivative. In the following we shall show how to solve for the shear modulus $\mu$ in the entire 3D volume. To this end, let's consider any plane parallel to the $x z$ plane, say $y=$ const. (see Figure 4.1). Following the same procedure as above, we can obtain two PDEs like Equation (4.32) and Equation (4.33), but containing $x$ - and $z$-derivatives, instead of $x$ - and $y$-derivatives:

$$
\begin{align*}
& a_{14}\left(\mu_{, x x}-\mu_{, z z}\right)+b_{14} \mu_{, x z}+e_{14} \mu_{, x}+f_{14} \mu_{, z}+h_{14} \mu+k_{14}=0  \tag{4.36}\\
& a_{15}\left(\mu_{, x x}-\mu_{, z z}\right)+b_{15} \mu_{, x z}+e_{15} \mu_{, x}+f_{15} \mu_{, z}+h_{15} \mu+k_{15}=0 . \tag{4.37}
\end{align*}
$$

Following the derivation in Section 3.1.2, these form a set of hyperbolic PDEs for $\boldsymbol{\mu}=\left[\mu^{r}, \mu^{i}\right]$. Thus with Cauchy data known on a curve $C$, in this case along
$y=$ const. $\& z=0$, we can determine $\mu$ anywhere in this plane. This Cauchy data is given by Equation (4.34) and Equation (4.35). Therefore, by translating the $y=$ const. plane along the $y$ direction we can fill up the entire space.


Figure 4.1: A construction to examine the uniqueness of the 3D timeharmonic viscoelasticity problem with two measured displacement fields. Given two measurements, we can obtain four equations containing $\mu_{, z}, \mu_{, x z}, \mu_{, y z}, \mu_{, z z}$, and two equations only involving $x$ - and $y$-derivatives. These two equations, when restricted to the $x y$ plane, have the same form of the PDEs for the 2D plane strain problem with two measurements considered in Section 3.1.2. According to the analysis in this section, up to eight real-valued constants are required to find a unique solution in the $x y$ plane. Then we can solve for its $z$ derivatives using Equation (4.35). Thereafter, we consider any plane parallel to the $x z$ plane. Similarly, we can obtain two equations like Equation (4.32) and Equation (4.33), but containing $x$ - and $z$-derivatives. Thus with Cauchy data known on the initial curve, in this case along $y=$ const. $\& z=0$, we can determine $\mu$ anywhere in this plane. Therefore, by translating this plane along $y$ direction we can fill up the entire space.

In summary, with two measured displacement fields that satisfy certain conditions of the strains, it is possible to determine $\mu$ in a 3D volume from up to eight pieces of real-valued data.

### 4.2 Weak Form: Complex Adjoint Weighted Equations

We note that the 3D time-harmonic viscoelasticity problem with one or more displacement fields satisfies the unified equation proposed in Section 3.1.3. Thus the complex adjoint weighted equations (CAWE) formulation developed for the unified equation and its properties work for the 3D problem. For completeness and easy referral, we state the CAWE formulation and the analysis of its properties here.

### 4.2.1 Problem Formulation

For multiple loadings, we define $\boldsymbol{\mu}=\left[\left(\mu, p^{(1)}, p^{(2)}, \cdots, p^{\left(N_{\text {loadings }}\right)}\right)\right]$ and $N_{\text {loadings }}$ indexes the total number of the loading conditions. Hence $\mu_{1}=\mu, \mu_{2}=p^{(1)}, \cdots$, $\mu_{N_{\text {loadings }}+1}=p^{\left(N_{\text {loadings }}\right)}$. The CAWE formulation for the 3D inverse problem with multiple measurements is given by: find $\boldsymbol{\mu} \in \mathcal{S}$ such that

$$
\begin{equation*}
b(\boldsymbol{w}, \boldsymbol{\mu})=l(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \mathcal{V} \tag{4.38}
\end{equation*}
$$

where

$$
\begin{align*}
b(\boldsymbol{w}, \boldsymbol{\mu}) & =\sum_{l=1}^{N_{\text {loadings }}} \sum_{m=1}^{N_{\text {params }}} b^{(l)}\left(w_{m}, \boldsymbol{\mu}\right)  \tag{4.39}\\
l(\boldsymbol{w}) & =\sum_{l=1}^{N_{\text {loadings }}} \sum_{m=1}^{N_{\text {params }}} l^{(l)}\left(w_{m}\right)  \tag{4.40}\\
b^{(l)}\left(w_{m}, \boldsymbol{\mu}\right) & =\left(\boldsymbol{A}^{(m, l)} \nabla w_{m}, \nabla \cdot \sum_{n=1}^{N_{\text {params }}}\left(\boldsymbol{A}^{(n, l)} \mu_{n}\right)\right)  \tag{4.41}\\
l^{(l)}\left(w_{m}\right) & =-\left(\boldsymbol{A}^{(m, l)} \nabla w_{m}, \boldsymbol{f}^{(l)}\right) \tag{4.42}
\end{align*}
$$

and

$$
\begin{equation*}
\boldsymbol{f}^{(l)}=\rho \omega^{(l) 2} \boldsymbol{u}^{(l)} . \tag{4.43}
\end{equation*}
$$

Here $l$ indexes the loading conditions, and $N_{\text {params }}=N_{\text {loadings }}+1$, is the number of parameters.

The weighting function space $\mathcal{V}$ and the trial solution space $\mathcal{S}$ are defined as

$$
\begin{align*}
\mathcal{V} & =\left\{(w, q) \mid w, q \in H^{1}(\Omega)\right\}  \tag{4.44}\\
\mathcal{S} & =\left\{(u, p) \mid u, p \in H^{1}(\Omega)\right\} \tag{4.45}
\end{align*}
$$

The space $\mathcal{V}$ and $\mathcal{S}$ differ from each other on the basis of the boundary conditions specified on $\mu$ and $p$. Since these may change depending on the problem type, they are not specified here.

For example, when $N_{\text {loadings }}=2$, we have $\boldsymbol{\mu}=\left(\mu, p^{(1)}, p^{(2)}\right)$, the corresponding weighting function $\boldsymbol{w}=\left(w, q^{(1)}, q^{(2)}\right)$. The CAWE for the problem is given by: find $\boldsymbol{\mu} \in \mathcal{S}$ such that

$$
\begin{equation*}
b(\boldsymbol{w}, \boldsymbol{\mu})=l(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \mathcal{V} \tag{4.46}
\end{equation*}
$$

Here

$$
\left.\begin{array}{rl}
b(\boldsymbol{w}, \boldsymbol{\mu}) & =\left(2 \boldsymbol{\epsilon}^{(1)} \nabla w-\mathbf{1} \nabla q^{(1)},\right. \\
& +\left(2 \boldsymbol{\epsilon}^{(2)} \nabla w-\mathbf{1} \nabla q^{(2)},\right. \\
& \nabla \cdot\left(2 \boldsymbol{\epsilon}^{(2)} \mu-\mathbf{1} p^{(2)}\right) \tag{4.48}
\end{array}\right) .
$$

### 4.2.2 Analysis of CAWE Formulation

We now make three assumptions on the measured data that determine the well-posedness of the CAWE formulation.
(i) The CAWE bilinear form provides a natural norm on $\mathcal{V}$, that we call the $A$ norm. Thus we define:

$$
\begin{align*}
\|\boldsymbol{w}\|_{A}^{2} & \equiv \sum_{m=1}^{N_{\text {params }}} \sum_{n=1}^{N_{\text {params }}} \sum_{l=1}^{N_{\text {loadings }}}\left(\boldsymbol{A}^{(m, l)} \nabla w_{m}, \boldsymbol{A}^{(n, l)} \nabla w_{n}\right)  \tag{4.49}\\
& \equiv(\nabla \boldsymbol{w}, \boldsymbol{A} \nabla \boldsymbol{w}) \geqslant 0 \quad \forall \boldsymbol{w} \in \mathcal{V}, \tag{4.50}
\end{align*}
$$

where $[\nabla \boldsymbol{w}]_{m i}=\frac{\partial w_{m}}{\partial x_{i}}$, and $\boldsymbol{A}_{m n j k}=\sum_{i} \sum_{l} A_{i j}^{*(m, l)} A_{i k}^{(n, l)}$.

To call this a norm, we assume that

$$
\begin{equation*}
\|\boldsymbol{w}\|_{A}^{2}=0 \Leftrightarrow \boldsymbol{w}=\mathbf{0} \quad \operatorname{in} \Omega \tag{4.51}
\end{equation*}
$$

This happens when $\boldsymbol{A}$ is positive definite and $\mathcal{V}$ is such that a constant $\boldsymbol{w}$ is ruled out. We note it is not easy to postulate conditions on $\boldsymbol{A}^{(m, l)}$ and hence $\boldsymbol{u}^{(n)}$, to determine when this will be the case. We also note that when $\boldsymbol{A}$ is positive definite, the $A$-norm is analogous to an $H^{1}$ semi-norm on $\mathcal{V}$, and equivalent to the $H^{1}$ norm on $\mathcal{V}$ in many practical cases.
(ii) Let

$$
\begin{equation*}
q(\boldsymbol{w})=\sum_{m=1}^{N_{\text {params }}} \sum_{n=1}^{N_{\text {params }}} \sum_{l=1}^{N_{\text {loadings }}}\left(w_{m} \nabla \cdot \boldsymbol{A}^{(m, l)}, w_{n} \nabla \cdot \boldsymbol{A}^{(n, l)}\right) . \tag{4.52}
\end{equation*}
$$

We assume that there exists a constant $C_{p}^{A}<\infty$ such that

$$
\begin{equation*}
q(\boldsymbol{w}) \leqslant C\|\boldsymbol{w}\|_{A}^{2} \quad \forall C \geqslant C_{p}^{A} \tag{4.53}
\end{equation*}
$$

This is a generalization of Poincare inequality and will hold for small $\nabla \cdot \boldsymbol{A}^{(m, l)}$.
(iii) We assume the $A$-norm is bounded by the $H^{1}$ norm on $\mathcal{V}$. That is, there exists a finite, positive constant $C$ satisfying

$$
\begin{equation*}
\|\boldsymbol{w}\|_{A} \leqslant C\|\boldsymbol{w}\|_{1} . \tag{4.54}
\end{equation*}
$$

This implies that the Largest eigenvalue of $\boldsymbol{A}$ is bounded. When the data is such that all the conditions above are satisfied we can prove the coercivity of the CAWE formulation.

CAWE Stability We now examine coercivity of $b(\cdot, \cdot)$ :

$$
\begin{align*}
b(\boldsymbol{w}, \boldsymbol{w}) & =\sum_{m=1}^{N_{\text {params }}} \sum_{n=1}^{N_{\text {params }}} \sum_{l=1}^{N_{\text {loadings }}}\left(\boldsymbol{A}^{(m, l)} \nabla w_{m}, \nabla \cdot\left(\boldsymbol{A}^{(n, l)} w_{n}\right)\right) \\
& =\|\boldsymbol{w}\|_{A}^{2}+\sum_{m=1}^{N_{\text {params }}} \sum_{n=1}^{N_{\text {params }}} \sum_{l=1}^{N_{\text {loadings }}}\left(\boldsymbol{A}^{(m, l)} \nabla w_{m},\left(\nabla \cdot \boldsymbol{A}^{(n, l)}\right) w_{n}\right) \cdot( \tag{4.55}
\end{align*}
$$

For any $\epsilon>0$

$$
\begin{equation*}
\sum_{m=1}^{N_{\text {params }}} \sum_{n=1}^{N_{\text {params }}} \sum_{l=1}^{N_{\text {loadings }}}\left(\boldsymbol{A}^{(m, l)} \nabla w_{m},\left(\nabla \cdot \boldsymbol{A}^{(n, l)}\right) w_{n}\right) \geqslant-\frac{\epsilon}{2}\|\boldsymbol{w}\|_{A}^{2}-\frac{1}{2 \epsilon} q(\boldsymbol{w}) . \tag{4.56}
\end{equation*}
$$

To prove the inequality above, let's look at

$$
\begin{equation*}
\left\|\epsilon^{1 / 2} \sum_{m=1}^{N_{\text {params }}} \boldsymbol{A}^{(m, l)} \nabla w_{m}+\epsilon^{-1 / 2}\left(\sum_{m=1}^{N_{\text {params }}} \nabla \cdot \boldsymbol{A}^{(m, l)}\right) w_{m}\right\|^{2} \geq 0 \tag{4.57}
\end{equation*}
$$

Setting $x^{(l)}=\sum_{m=1}^{N_{\text {params }}} \boldsymbol{A}^{(m, l)} \nabla w_{m}$ and $y^{(l)}=\left(\sum_{m=1}^{N_{\text {params }}} \nabla \cdot \boldsymbol{A}^{(m, l)}\right) w_{m}$ arrives at

$$
\begin{align*}
& \left\|\epsilon^{1 / 2} x^{(l)}+\epsilon^{-1 / 2} y^{(l)}\right\|^{2} \geq 0  \tag{4.58}\\
\Rightarrow \quad & \left(\epsilon^{1 / 2} x^{(l)}+\epsilon^{-1 / 2} y^{(l)}, \epsilon^{1 / 2} x^{(l)}+\epsilon^{-1 / 2} y^{(l)}\right) \geq 0  \tag{4.59}\\
\Rightarrow \quad & \left\|\epsilon^{1 / 2} x^{(l)}\right\|^{2}+\left\|\epsilon^{-1 / 2} y^{(l)}\right\|^{2}+2\left(x^{(l)}, y^{(l)}\right) \geq 0  \tag{4.60}\\
\Rightarrow \quad & \left(x^{(l)}, y^{(l)}\right) \geq-\frac{\epsilon}{2}\left\|x^{(l)}\right\|^{2}-\frac{1}{2 \epsilon}\left\|y^{(l)}\right\|^{2} . \tag{4.61}
\end{align*}
$$

Summing over $l$,

$$
\begin{equation*}
\sum_{l}\left(x^{(l)}, y^{(l)}\right) \geq-\frac{\epsilon}{2} \sum_{l}\left\|x^{(l)}\right\|^{2}-\frac{1}{2 \epsilon} \sum_{l}\left\|y^{(l)}\right\|^{2} \tag{4.62}
\end{equation*}
$$

Then we have the desired result.

Using (4.56) in (4.55) gives

$$
\begin{align*}
b(\boldsymbol{w}, \boldsymbol{w}) & \geqslant\left(1-\frac{\epsilon}{2}\right)\|\boldsymbol{w}\|_{A}^{2}-\frac{1}{2 \epsilon} q(\boldsymbol{w})  \tag{4.63}\\
& \geqslant\left[1-\frac{\epsilon}{2}-\frac{1}{2 \epsilon} C_{p}^{A}\right]\|\boldsymbol{w}\|_{A}^{2}  \tag{4.64}\\
& =C_{1}\|\boldsymbol{w}\|_{A}^{2} \tag{4.65}
\end{align*}
$$

where $C_{1}$ is in the brackets in (4.64); setting $\epsilon=\sqrt{C_{p}^{A}}$ gives

$$
\begin{equation*}
C_{1}=1-\sqrt{C_{p}^{A}} \tag{4.66}
\end{equation*}
$$

Note, we have stability for $C_{p}^{A}<1$.

CAWE Uniqueness Suppose $\boldsymbol{\mu}_{1}$ and $\boldsymbol{\mu}_{2}$ both satisfy Equation (4.38). Let $\boldsymbol{v}=$ $\boldsymbol{\mu}_{1}-\boldsymbol{\mu}_{2} \in \mathcal{V}$. Then according to the bi-linearity of $b(\cdot, \cdot)$, we have

$$
\begin{equation*}
b(\boldsymbol{w}, \boldsymbol{v})=0 \quad \forall \boldsymbol{w} \in \mathcal{V} \tag{4.67}
\end{equation*}
$$

Since $\boldsymbol{v} \in \mathcal{V}$, we may choose $\boldsymbol{w}=\boldsymbol{v}$ to find $b(\boldsymbol{v}, \boldsymbol{v})=0$. By the coercivity of $b(\cdot, \cdot)$, we conclude that $\boldsymbol{v}=\mathbf{0}$, and hence $\boldsymbol{\mu}_{1}=\boldsymbol{\mu}_{2}$. That is, the solution of Equation (4.38) is unique.

### 4.3 Numerical Approximation

We use a straightforward finite element discretization of the CAWE formulation. We approximate the infinite dimensional spaces by their finite dimensional counterparts $\mathcal{V}^{h} \subset \mathcal{V}$ and $\mathcal{S}^{h} \subset \mathcal{S}$ by using the standard piecewise constant finite element shape functions. The statement of this method is: find $\boldsymbol{\mu}^{h} \in \mathcal{S}^{h}$ such that

$$
\begin{equation*}
b\left(\boldsymbol{w}^{h}, \boldsymbol{\mu}^{h}\right)=l\left(\boldsymbol{w}^{h}\right) \quad \forall \boldsymbol{w}^{h} \in \mathcal{V}^{h} \tag{4.68}
\end{equation*}
$$

Since $\mathcal{S}^{h} \subset \mathcal{S}$ the continuous solution $\mu$ also satisfies (4.68). That is

$$
\begin{equation*}
b\left(\boldsymbol{w}^{h}, \boldsymbol{\mu}\right)=l\left(\boldsymbol{w}^{h}\right) \quad \forall \boldsymbol{w}^{h} \in \mathcal{V}^{h} \tag{4.69}
\end{equation*}
$$

Next we prove that our numerical solution converges at optimal rates to the exact solution under the restrictions of Section 4.2. Denote the Galerkin discretization error by $\boldsymbol{e}=\boldsymbol{\mu}-\boldsymbol{\mu}^{h}$. Then $\boldsymbol{e}$ satisfies

$$
\begin{equation*}
b\left(\boldsymbol{w}^{h}, \boldsymbol{e}\right)=0 \quad \forall \boldsymbol{w}^{h} \in \mathcal{V}^{h} \tag{4.70}
\end{equation*}
$$

We split the error $\boldsymbol{e}=\boldsymbol{\eta}+\boldsymbol{e}^{h}$, where $\boldsymbol{\eta}=\boldsymbol{\mu}-\boldsymbol{\mu}^{i}$ and $\boldsymbol{e}^{h}=\boldsymbol{\mu}^{i}-\boldsymbol{\mu}^{h}$. Here $\boldsymbol{\mu}^{i}$ is the best approximation to $\mu$ in the space $\mathcal{V}^{h}, \boldsymbol{\eta}$ is the interpolation error. Then by linearity of $b(\cdot, \cdot)$ we have:

$$
\begin{equation*}
b\left(\boldsymbol{w}^{h}, \boldsymbol{e}\right)=b\left(\boldsymbol{w}^{h}, \boldsymbol{\eta}+\boldsymbol{e}^{h}\right)=b\left(\boldsymbol{w}^{h}, \boldsymbol{\eta}\right)+b\left(\boldsymbol{w}^{h}, \boldsymbol{e}^{h}\right)=0 \quad \forall \boldsymbol{w}^{h} \in \mathcal{V}^{h} \tag{4.71}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\left|b\left(\boldsymbol{w}^{h}, \boldsymbol{e}^{h}\right)\right|=\left|b\left(\boldsymbol{w}^{h}, \boldsymbol{\eta}\right)\right| \quad \forall \boldsymbol{w}^{h} \in \mathcal{V}^{h} \tag{4.72}
\end{equation*}
$$

Select $\boldsymbol{w}^{h}=\boldsymbol{e}^{h}$ in the equation above to get:

$$
\begin{equation*}
b\left(\boldsymbol{e}^{h}, \boldsymbol{e}^{h}\right)=\left|b\left(\boldsymbol{e}^{h}, \boldsymbol{\eta}\right)\right| \tag{4.73}
\end{equation*}
$$

By continuity of $b(\cdot, \cdot)$ [c.f.[33]], we have

$$
\begin{equation*}
\left|b\left(\boldsymbol{e}^{h}, \boldsymbol{\eta}\right)\right| \leqslant C_{2}\left\|\boldsymbol{e}^{h}\right\|_{A}\|\boldsymbol{\eta}\|_{A} \tag{4.74}
\end{equation*}
$$

By coercivity of $b(\cdot, \cdot)$, we have

$$
\begin{equation*}
b\left(\boldsymbol{e}^{h}, \boldsymbol{e}^{h}\right) \geqslant C_{1}\left\|\boldsymbol{e}^{h}\right\|_{A}^{2} . \tag{4.75}
\end{equation*}
$$

Equations (4.73) - (4.74) give

$$
\begin{equation*}
C_{1}\left\|\boldsymbol{e}^{h}\right\|_{A}^{2} \leqslant C_{2}\left\|\boldsymbol{e}^{h}\right\|_{A}\|\boldsymbol{\eta}\|_{A} \tag{4.76}
\end{equation*}
$$

Therefore,

$$
\begin{array}{rll}
\|\boldsymbol{e}\|_{A} & =\left\|\boldsymbol{\eta}+\boldsymbol{e}^{h}\right\|_{A} & \\
& \leqslant\|\boldsymbol{\eta}\|_{A}+\left\|\boldsymbol{e}^{h}\right\|_{A} & \text { triangle inequality } \\
& \leqslant\left(1+\frac{C_{2}}{C_{1}}\right)\|\boldsymbol{\eta}\|_{A} \quad \text { by }(4.76) \\
& \leqslant\left(1+\frac{C_{2}}{C_{1}}\right) C\|\boldsymbol{\eta}\|_{1} \quad \text { by }(4.54) \\
& \leqslant C_{3} h^{p} . \quad \text { by interpolation estimate } \tag{4.81}
\end{array}
$$

Here $h$ represents the element size and $p$ is the polynomial order of completeness of functions in $\mathcal{V}^{h}$.

### 4.3.1 MR Measured Data

We now test the performance of the CAWE for the three-dimensional inverse viscoelasticity problem using a displacement field measured using MRI. This data was collected in a tissue-mimicking gelatin phantom and the experiment was conducted at the Mayo clinic [42, 43]. The experimental setup is described in detail in Section 2.4.3. For the three-dimensional data the displacements were measured at four instances, hence the dimension of the data is $256 \times 256 \times 16 \times 4(y \times x \times z \times t)$ with a mesh size of $0.6275 \mathrm{~mm} \times 0.6275 \mathrm{~mm} \times 2 \mathrm{~mm}(d y \times d x \times d z)$. In each xy plane no signal exists near the border and thus we work with a reduced image with dimensions $200 \times 160$ pixels $(y \times x)$. In the z-direction we choose the five central imaging planes, from the fourth to the eighth, among the 16 planes available. This choice is motivated by the fact that the quality of the outer planes is not good enough to use. In the outer planes the signal does not exist or part of them exist. These seven planes are centered about the 6 th plane, which is also the center of the phantom in the experimental setup. Therefore, the final dimensions of the domain of interest are $200 \times 160 \times 5 \times 4$ pixels $(y \times x \times z \times t)$.

We take the Fourier transform of the displacement data into the frequency domain and extract displacement at the driving frequency, $\omega=2 \pi \times 300 \mathrm{rad} / \mathrm{s}$. Then we scale the $x, y, z$-displacement components by the maximum magnitude of the dominant displacement component, that is the $z$-component, in the volume of
interest such that the maximum magnitude of the dominant component is 1 . The density of the gelatin is $\rho=10^{-3} \mathrm{~g} / \mathrm{mm}^{3}$.

Next, we smooth the displacement data in each imaging plane using a quadratic least squares filter that was used for the anti-plane shear case in Section 2.4.3. We do not use a three-dimensional quadratic least squares filter to smooth the displacement data in the entire 3D volume, since we find that the window size in $z$ direction cannot be larger than one. Otherwise some of the detail of the phantom is lost. Hence, we choose a window size of $5 \times 5 \times 1(y \times x \times z)$. Once the smooth have been determined, we evaluate the $x$ and $y$ derivatives (needed to compute strains) by differentiating the expression for the filtered field. For the $z$ derivative we use a mid point rule. Since this filter is a "centered" type of filter we do not apply it to the two pixels nearest to any edge in the $x y$ plane. Also, by evaluating derivatives in the $z$ - direction using a central difference formula we loose two planes in this direction. As a result our final imaging domain is $196 \times 156 \times 3(y \times x \times z)$.

We prescribe the boundary data for the shear modulus on surfaces that are parallel to the $z$ axis to be $20 .+i 0.5 K P a$. This value is estimated by fitting a plane wave to the z-component of the displacement in the homogeneous region of one of the $x-y$ planes. This is described in detail in Section 2.4.3. We also fix the value of the pressure at the origin to get a unique distribution of the pressure.

We utilize the boundary data, and the smoothed displacement and strain fields in the CAWE formulation to recover the complex-valued shear modulus. In particular, we consider imposing boundary data weakly through penalty terms. Furthermore, we append to the CAWE formulation the total variation diminishing (TVD) regularization to improve the performance of the CAWE formulation in the presence of noise. In order to simplify the notation we suppress the superscript $h$. The final
version of our formulation is : Find $\boldsymbol{\mu} \in \mathcal{S}$ such that

$$
\begin{align*}
b(\boldsymbol{w}, \boldsymbol{\mu}) & +\gamma_{1} \operatorname{Re}\left\{\left(w_{1}, \mu-\bar{\mu}\right)_{\Gamma_{g}}\right\} \\
& +\gamma_{2} \operatorname{Re}\left\{w_{2}\left(\boldsymbol{x}_{0}\right)\left(p^{(1) h}\left(\boldsymbol{x}_{0}\right)-\bar{p}^{(1) h}\left(\boldsymbol{x}_{0}\right)\right)\right\} \\
& +\gamma_{3} \operatorname{Re}\left\{w_{3}\left(\boldsymbol{x}_{0}\right)\left(p^{(2) h}\left(\boldsymbol{x}_{0}\right)-\bar{p}^{(2) h}\left(\boldsymbol{x}_{0}\right)\right)\right\} \\
& +\alpha_{1} \operatorname{Re}\left\{\left(\nabla w_{1}^{r}, \frac{\nabla \mu^{r}}{\sqrt{\left|\nabla \mu^{r}\right|^{2}+\beta^{2}}}\right)\right\} \\
& +\alpha_{2} \operatorname{Re}\left\{\left(\nabla w_{1}^{i}, \frac{\nabla \mu^{i}}{\sqrt{\left|\nabla \mu^{i}\right|^{2}+\beta^{2}}}\right)\right\} \\
& +\alpha_{3} \operatorname{Re}\left\{\left(\nabla w_{2}^{r}, \frac{\nabla p^{(1) r}}{\sqrt{\left|\nabla p^{(1) r}\right|^{2}+\beta^{2}}}\right)\right\} \\
& +\alpha_{4} \operatorname{Re}\left\{\left(\nabla w_{2}^{i}, \frac{\nabla p^{(1) i}}{\sqrt{\left|\nabla p^{(1) i}\right|^{2}+\beta^{2}}}\right)\right\} \\
& +\alpha_{5} \operatorname{Re}\left\{\left(\nabla w_{3}^{r}, \frac{\nabla p^{(2) r}}{\sqrt{\left|\nabla p^{(2) r}\right|^{2}+\beta^{2}}}\right)\right\} \\
& +\alpha_{6} \operatorname{Re}\left\{\left(\nabla w_{3}^{i}, \frac{\nabla p^{(2) i}}{\sqrt{\left|\nabla p^{(2) i}\right|^{2}+\beta^{2}}}\right)\right\} \\
& =l(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \mathcal{V}, \tag{4.82}
\end{align*}
$$

where $\alpha_{j}$ are the regularization parameters, $\beta$ is a parameter selected to ensure that the regularization term is continuous when $\nabla \mu^{r}=\mathbf{0}, \nabla \mu^{i}=\mathbf{0}, \nabla p^{(1) r}=\mathbf{0}$, $\nabla p^{(1) i}=\mathbf{0}, \nabla p^{(2) r}=\mathbf{0}$ or $\nabla p^{(2) i}=\mathbf{0} . \gamma_{j}$ are penalty parameters for imposing the weak boundary conditions, and $(\cdot, \cdot)_{\Gamma_{g}}$ denotes the $L_{2}$ inner product over the portion of the boundary where data for $\mu$ is prescribed. The reconstruction was performed on mesh of regular hexahedral elements with the same grid points that were used for measuring displacements. The displacements and strains were interpolated using the standard $C^{0}$ finite element shape functions, while the derivatives of strains were evaluated by differentiating the strains within each element. Only the real part of the shear modulus was recovered since the imaginary part of the shear modulus is much smaller compared to the real part of the shear modulus.

Figure 4.2 shows the displacement components in the volume of interest at the driving frequency. From left to right, we show the displacement components in the 5th, 6th and 7th imaging $(x y)$ planes. From top to bottom we show displace-
ment components in the $x, y$ and $z$ directions. We observe that the displacement components display very little variation and that the dominant component in the z direction, which coincides with the direction of external excitation. Given this, and the fact that the material properties of the phantom do not vary in the z-direction, we note that an "out-of-plane shear" state is a good approximation for this problem. This was seen in Chapter 2, where reasonable accurate reconstructions were obtained while making use of this assumption. We also note that in the $z$ plots the effect of an inhomogeneity located at toward the top-center region of the phantom is clearly seen.

Figure 4.3 - Figure 4.5 show the reconstructions of the real part of the shear modulus in the three-dimensional volume. We present the results in each imaging plane. We observe that the shape and the location of the large inclusion in each imaging plane is recovered. The value of the shear modulus in the large inclusion is also well estimated. The experimental value is reported to be $130 K P a$ in the inclusion and $20 K P a$ in the background. However in Figure 4.4 and Figure 4.5 the results are dominated by artifacts making the detection of the small inclusion difficult. This inclusion is supposed to be located at toward the bottom-center region of the region as shown in Figure 4.3, and has a shear modulus value equal to $130 K P a$. In Figure 4.3, although significant artifacts are also present, especially near the large inclusion, the small inclusion is visible. The exact contrast in the real part of the shear modulus between the small inclusion and the background is $6.5: 1$, while in the reconstruction the contrast is underestimated and is around $2.5: 1$. We also notice that the artifacts in all the images are in the form of wavy variations and the amplitude of some variations is about the same as the value of the real part of the shear modulus. This is indicative of a high level of noise. Further, recognizing that for the same problem in Section 2.4.3 we were able to produce much more accurate reconstructions while making use of data only in the $x y$ plane, we believe that the dominant source of noise is the calculation of strains in the z-direction. To accomplish this we have relied on a simple midpoint rule, and have utilized only three imaging planes. This could be improved by including more imaging planes and a more accurate approach to estimating strains. Another way
to improve the reconstruction is to impose incompressibility when we smooth the displacement data.

### 4.4 Chapter Summary

We have considered the inverse problem of three-dimensional time-harmonic incompressible isotropic viscoelasticity. We have characterized this problem and proved the uniqueness of its solution in three situations. We conclude that the data required to obtain a unique solution depends on the number of available measurements. When a single displacement field is given, the problem is elliptic in most cases but can be hyperbolic in some select cases. The use of multiple measurements reduces the need for boundary data significantly. When two displacement fields are available, only eight real-valued constants are needed to determine the complex valued shear modulus uniquely, and two real-valued constants for each pressure field are required to estimate the complex valued pressure uniquely.

In order to solve this inverse problem, we extended the CAWE formulation developed in Section 3.2 to the three-dimensional time-harmonic viscoelasticity problem. We appended the total variation diminishing (TVD) regularization to the CAWE formulation to in order to handle the effect of noise while preserving the sharpness of changes at the interface of two different materials.

We considered a simple finite element discretization of the regularized CAWE formulation and tested the performance of this algorithm on experimental data. The "exact" solution comprised of two inclusion of diameters 16 mm and 3 mm and shear modulus 130 KPa , respectively, embedded in a homogeneous material with a shear modulus of $20 K P a$. The algorithm performed well in detecting the location and shape of the large inclusion. The value of the shear modulus in the large inclusion was also estimated well. However significant artifacts present in the result. These made the detection of the smaller inclusion harder. When compared with the reconstructions for the out-of-plane case in Chapter 2, this indicated that the source of these errors was the inaccurate calculation for the strains in the z-direction.


Figure 4.2: 3D displacement data with the dimension of $196 \times 156 \times 3(y \times$ $x \times z)$. Left: 5th imaging plane; Middle: 6th imaging plane; Right: 7th imaging plane. Top: $x$ component; Middle: $y$ component; Bottom: $z$ component.

(a)

Figure 4.3: Reconstruction of the real part of the shear modulus in the 5th plane from MR measured data using CAWE ( $\alpha_{1}=$ $\left.1.9 e 8, \alpha_{2}=5 e 11, \alpha_{j}=0(j=3, \cdots, 6), \beta=1.0, \gamma_{j}=1.0 e 5\right)$.

(a)

Figure 4.4: Reconstruction of the real part of the shear modulus in the 6 th plane from MR measured data using CAWE ( $\alpha_{1}=$ $\left.1.9 e 8, \alpha_{2}=5 e 11, \alpha_{j}=0(j=3, \cdots, 6), \beta=1.0, \gamma_{j}=1.0 e 5\right)$.

(a)

Figure 4.5: Reconstruction of the real part of the shear modulus in the 7 th plane from measured MR data using CAWE ( $\alpha_{1}=$ $\left.1.9 e 8, \alpha_{2}=5 e 11, \alpha_{j}=0(j=3, \cdots, 6), \beta=1.0, \gamma_{j}=1.0 e 5\right)$.

## CHAPTER 5

## Conclusions

In this dissertation, we have considered the inverse problem of determining the spatial distribution of the complex-valued shear modulus within an incompressible linear viscoelastic solid undergoing infinitesimal, time-harmonic deformation, from the knowledge of one and two displacement fields in its interior.

The governing equations of the inverse problem are the 3D time-harmonic viscoelastic equations. We started our work in Chapter 2, studying on a simplified mathematical model, named the scalar Helmholtz equation. The two-dimensional problems of anti-plane shear with two displacement fields and plane stress with a single displacement field are required to satisfy two independent scalar Helmholtz equations. We have analyzed the strong form of this simplified mathematical model and found that it requires relatively strong restrictions on measured data which are challenges for noisy measurements. We have addressed this issue by developing a novel, weak formulation of the original partial differential equations (PDE), which is obtained by weighting the original PDE by its adjoint operator acting on the complex-conjugate of the weighting function. We termed this formulation the complex adjoint weighted equation (CAWE). We have proved that the solutions of the CAWE formulation exists and is unique under much milder conditions on the measured data. We then implemented a simple, straightforward discretization of the CAWE formulation and tested it on synthetically generated data and experimentally, either ultrasound-measured or magnetic resonance-measured data. In this regard, we have found that it is less diffusive than the corresponding least squares weak formulation and can recover the complex-valued shear modulus fairly well.

In Chapter 3, we have considered the inverse problem of plane strain. The assumptions leading the 3D time-harmonic viscoelasticity to this simplified mathematical model are different compared to the scalar Helmholtz model. In the plane strain case, the pressure term remains in the equation and is an unknown to recover, while in the scalar Helmholtz model the pressure is either eliminated or ignored. We
have examined the uniqueness of this problem. We concluded that with a single displacement field, the type of the problem depends on the strain field. It could be hyperbolic or elliptic. Further, we have investigated the inverse problem of plane strain with two measurements and have found that the need for boundary data is significantly reduced and the dimension of the boundary data is only eight for the complex-valued shear modulus. Thereafter We described a unified equation for one ore more measurements and proposed the CAWE formulation of this unified equation. We have proved that the CAWE formulation is well-posedness under some conditions on the measured data. We have examined performance of the CAWE formulation by solving the inverse problem, where the displacement data was generated by Abaqus and the complex-valued shear modulus was sought. Then we added Gaussian white noise to the synthetic displacement data such that the strains were of $20 \%$ noise and then tested the formulation on it. We have considered appending to the CAWE formulation the total variation diminishing to improve its performance in the presence of noise. We have found that the regularized CAWE formulation has good ability to detect the inclusion with reasonably accurate values.

In Chapter 4, we have extended the CAWE formulation to the three dimensional time-harmonic viscoelasticity problems. We have analyzed the problem and concluded that with a single measurement the problem could be either an elliptic problem or a hyperbolic problem. We have found that with two measurements the problem changes to be purely hyperbolic. We have examined the uniqueness of the three dimensional time-harmonic viscoelasticity problem and found that with two measurements, the dimension of the boundary data to obtain a unique solution for the shear modulus is eight. We have presented and validated the CAWE formulation of this problem by solving for the shear modulus from magnetic resonance data. We have considered the use of the TVD regularization and have observed that the regularized CAWE formulation performs well in detecting the shape and the location of the large inclusion as well as the value of the real part of the shear modulus in the large inclusion. We have also recovered the small inclusion, which is of 3 mm diameter. However the significant artifacts in the result makes the small inclusion undetectable. We have analyzed it and attributed it to the inaccurate calculation
for the strains in the $z$ direction.
Future work comprises the application of the CAWE formulation on more experimental data with different experimental setups and imaging techniques. Furthermore, it is still a challenge to reconstruct the imaginary part of the shear modulus once the ratio of the imaginary part of the shear modulus to the real part of the shear modulus is very low.

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## APPENDIX A Compatibility Conditions of Two Displacement Fields

Equations for $\mu$ are

$$
\begin{equation*}
\boldsymbol{a}^{(i)} \cdot \nabla \mu+\mu \nabla \cdot \boldsymbol{a}^{(i)}+f^{(i)}=0, \quad i=1,2 . \tag{A.1}
\end{equation*}
$$

Multiplying these by $\boldsymbol{a}^{(i) *}$ and adding the resulting equations we arrive at

$$
\begin{equation*}
\boldsymbol{A} \cdot \nabla \mu+\boldsymbol{a} \mu+\boldsymbol{f}=\mathbf{0} \tag{A.2}
\end{equation*}
$$

The solution to this equation is given by the sum of a homogeneous and a particular part $\mu=\mu^{h}+\mu^{p}$, where the equation for $\mu^{h}$ is

$$
\begin{equation*}
\nabla \mu^{h}+\boldsymbol{A}^{-1} \boldsymbol{a} \mu^{h}=\mathbf{0} . \tag{A.3}
\end{equation*}
$$

We write $\mu^{p}=\mu^{h} g$, which yields the following equation for $g$,

$$
\begin{equation*}
\nabla g+\frac{\boldsymbol{A}^{-1} \boldsymbol{f}}{\mu^{h}}=\mathbf{0} . \tag{A.4}
\end{equation*}
$$

The solution to (A.3) and (A.4) yields

$$
\begin{align*}
& \mu^{h}(\boldsymbol{x})=\mu_{0} \exp \left(-\int_{\boldsymbol{x}_{p}}^{\boldsymbol{x}} \boldsymbol{A}^{-1}\left(\boldsymbol{x}^{\prime}\right) \boldsymbol{a}\left(\boldsymbol{x}^{\prime}\right) \cdot d \boldsymbol{x}^{\prime}\right)  \tag{A.5}\\
& \mu^{p}(\boldsymbol{x})=-\mu^{h}(\boldsymbol{x}) \int_{\boldsymbol{x}_{p}}^{\boldsymbol{x}} \frac{\boldsymbol{A}^{-1}\left(\boldsymbol{x}^{\prime}\right) \boldsymbol{f}\left(\boldsymbol{x}^{\prime}\right)}{\mu^{h}\left(\boldsymbol{x}^{\prime}\right)} \cdot d \boldsymbol{x}^{\prime} \tag{A.6}
\end{align*}
$$

Taking the curl of (A.3) yields the compatibility condition for $\mu^{h}$ to exist, viz.,

$$
\begin{equation*}
\nabla \times\left(\boldsymbol{A}^{-1} \boldsymbol{a}\right)=0 \tag{A.7}
\end{equation*}
$$

Taking the curl of (A.4) and eliminating $\mu^{h}$ using (A.3) yields the following
compatibility condition for $\mu^{p}$ to exist

$$
\begin{equation*}
\boldsymbol{C}: \nabla\left(\boldsymbol{A}^{-1} \boldsymbol{f}\right)+\left(\boldsymbol{A}^{-1} \boldsymbol{f}\right) \cdot \boldsymbol{C}\left(\boldsymbol{A}^{-1} \boldsymbol{a}\right)=0 \tag{A.8}
\end{equation*}
$$

where $\boldsymbol{C}=\left[\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right]$.

## APPENDIX B Uniqueness Proof for Hyperbolic System of Second Order

In the following we prove the uniqueness of the solution to the second-order hyperbolic system

$$
\begin{align*}
& u_{, y y}-u_{, x x}-a_{1}^{\prime} u_{, y}-b_{1}^{\prime} u_{, x}-c_{1}^{\prime} u-d_{1}^{\prime} v_{, y}-e_{1}^{\prime} v_{, x}-f_{1}^{\prime} v=0  \tag{B.1}\\
& v_{, y y}-v_{, x x}-a_{2}^{\prime} u_{, y}-b_{2}^{\prime} u_{, x}-c_{2}^{\prime} u-d_{2}^{\prime} v_{, y}-e_{2}^{\prime} v_{, x}-f_{2}^{\prime} v=0 . \tag{B.2}
\end{align*}
$$

Let $\gamma$ be a triangle region of the $x y$ plane bounded by an initial curve $A B$ which is nowhere characteristic, and by the two characteristic lines $P A(x-y=$ const. $)$ and $P B(x+y=$ const.) (Figure B.1). Our object is to show: in the hyperbolic system (B.1) and (B.2), if $u, v$ and their derivatives $u_{, x}, u_{, y}, v_{, x}, v_{, y}$ vanish on $A B$, then $u, v$ vanish identically in the entire region $\Gamma$. To this end we cut off the vertex of our triangle by means of a straight line $y=$ const., which intersects $P A$ and $P B$ in the points $C$ and $D$, obtaining a smaller triangle whose base we denote by $H_{y}$, and a trapezoid $\Gamma_{y}$. When $y=h, H_{y}$ is denoted as $H_{h}$, and $\Gamma_{y}$ is $\Gamma_{h}$.

We first note that any point $(x, y)$ in $\Gamma$,

$$
\begin{align*}
& u(x, y)=\int_{0}^{y} u_{\tau}(x, \tau) d \tau  \tag{B.3}\\
& v(x, y)=\int_{0}^{y} v_{\tau}(x, \tau) d \tau \tag{B.4}
\end{align*}
$$

hence, by Schwarz's inequality,

$$
\begin{align*}
u^{2}(x, y) & =\left(\int_{0}^{y} u_{\tau}(x, \tau) d \tau\right)^{2}  \tag{B.5}\\
& =\left(\int_{0}^{y}\left(u_{\tau}(x, \tau) \cdot 1\right) d \tau\right)^{2} \\
& \leq \int_{0}^{y} u_{\tau}^{2} d \tau \int_{0}^{y} 1^{2} d \tau \\
& =y \int_{0}^{y} u_{\tau}^{2} d \tau
\end{align*}
$$



Figure B.1: A construction to show that in the hyperbolic system of second order (B.1) and (B.2), if $u, v$ and their derivatives $u_{, x}, u_{, y}, v_{, x}, v_{, y}$ vanish on $A B$, then $u, v$ vanish identically in the entire region $\Gamma . A B$ is an initial curve. $P A$ and $P B$ are the characteristic lines.
and similarly

$$
\begin{equation*}
v^{2}(x, y) \leq y \int_{0}^{y} v_{\tau}^{2}(x, \tau) d \tau \tag{B.6}
\end{equation*}
$$

Integrating over $H_{y}$, we have

$$
\begin{align*}
\int_{H_{y}} u^{2}(x, y) d x & \leq \iint_{\Gamma_{H_{y}}} y u_{\tau}^{2}(x, \tau) d x d \tau \\
& \leq y \iint_{\Gamma_{y}} u_{\tau}^{2}(x, \tau) d x d \tau \tag{B.7}
\end{align*}
$$

here $\Gamma_{H_{y}}$ is the rectangle as shown in the figure. Its area is less than the area of the trapzoid $\Gamma_{y}$. Since the integrand is non-negative, the second inequality is true. Similarly

$$
\begin{equation*}
\int_{H_{y}} v^{2}(x, y) d x \leq y \iint_{\Gamma_{y}} v_{\tau}^{2}(x, \tau) d x d \tau . \tag{B.8}
\end{equation*}
$$

Integrating (B.7) from $y=0$ to $y=h$, we have

$$
\begin{align*}
\iint_{\Gamma_{h}} u^{2}(x, y) d x d y & \leq \int_{0}^{h}\left[y \iint_{\Gamma_{y}} u_{\tau}^{2}(x, \tau) d x d \tau\right] d y  \tag{B.9}\\
& \leq \int_{0}^{h}\left[h \iint_{\Gamma_{h}} u_{\tau}^{2}(x, \tau) d x d \tau\right] d y  \tag{B.10}\\
& \leq h^{2} \iint_{\Gamma_{h}} u_{\tau}^{2}(x, \tau) d x d \tau  \tag{B.11}\\
& \leq h^{2} \iint_{\Gamma_{h}}\left(u_{\tau}^{2}+u_{x}^{2}\right) d x d \tau \tag{B.12}
\end{align*}
$$

Inequality (B.10) is true because $y \leq h$, the region of $\Gamma_{y}$ is less than the region of $\Gamma_{h}$, and the integrand is non-negative. Inequality (B.12) comes out by adding one non-negative term $u_{x}^{2}$ into the integrand. Similarly

$$
\begin{equation*}
\iint_{\Gamma_{h}} v^{2}(x, y) d x d y \leq h^{2} \iint_{\Gamma_{h}}\left(v_{\tau}^{2}+v_{x}^{2}\right) d x d \tau \tag{B.13}
\end{equation*}
$$

Now define an 'energy integral'

$$
\begin{equation*}
E(h)=\int_{H_{h}}\left(u_{, x}^{2}+u_{, y}^{2}\right)+\left(v_{, x}^{2}+v_{, y}^{2}\right) d x \tag{B.14}
\end{equation*}
$$

and integrate the following indentity over $\Gamma_{h}$

$$
\begin{align*}
0= & 2\left(\boldsymbol{v}_{, y}, L[\boldsymbol{v}]\right)  \tag{B.15}\\
= & 2 u_{, y}\left(u_{, y y}-u_{, x x}-a_{1}^{\prime} u_{, y}-b_{1}^{\prime} u_{, x}-c_{1}^{\prime} u-d_{1}^{\prime} v_{, y}-e_{1}^{\prime} v_{, x}-f_{1}^{\prime} v\right) \\
& +2 v_{, y}\left(v_{, y y}-v_{, x x}-a_{2}^{\prime} u_{, y}-b_{2}^{\prime} u_{, x}-c_{2}^{\prime} u-d_{2}^{\prime} v_{, y}-e_{2}^{\prime} v_{, x}-f_{2}^{\prime} v\right),
\end{align*}
$$

in which $\boldsymbol{v}=(u, v)$ and $L[\boldsymbol{v}]$ is the equation system (B.1) and (B.2). Since

$$
\begin{aligned}
2 u_{, y}\left(u_{, y y}-u_{, x x}\right) & =2 u_{, y} u_{, y y}-2 u_{, y} u_{, x x} \\
& =\left(u_{, y}^{2}\right)_{, y}+\left(u_{, x}^{2}\right)_{, y}-\left(2 u_{, x} u_{, y}\right)_{, x},
\end{aligned}
$$

the equation (B.15) arrives at

$$
\begin{align*}
0= & \left(u_{, x}^{2}+u_{, y}^{2}\right)_{, y}-2\left(u_{, x} u_{, y}\right)_{, x}-2 a_{1}^{\prime} u_{, y}^{2}-2 b_{1}^{\prime} u_{, x} u_{, y}  \tag{B.16}\\
& -2 c_{1}^{\prime} u u_{, y}-2 d_{1}^{\prime} v_{, y} u_{, y}-2 e_{1}^{\prime} v_{, x} u_{, y}-2 f_{1}^{\prime} v u_{, y} \\
& +\left(v_{, x}^{2}+v_{, y}^{2}\right)_{, y}-2\left(v_{, x} v_{, y}\right)_{, x}-2 d_{2}^{\prime} v_{, y}^{2}-2 e_{2}^{\prime} v_{, x} v_{, y} \\
& -2 f_{2}^{\prime} v v_{, y}-2 a_{2}^{\prime} u_{, y} v_{, y}-2 b_{2}^{\prime} u_{, x} v_{, y}-2 c_{2}^{\prime} u v_{, y} .
\end{align*}
$$

Integrate over the region $\Gamma_{h}$,

$$
\begin{align*}
& \iint_{\Gamma_{h}}\left(\left(u_{, x}^{2}+u_{, y}^{2}\right)_{, y}-2\left(u_{, x} u_{, y}\right)_{, x}\right)+\left(\left(v_{, x}^{2}+v_{, y}^{2}\right)_{, y}-2\left(v_{, x} v_{, y}\right)_{, x}\right) d x d y  \tag{B.17}\\
= & \iint_{\Gamma_{h}}\left(2 a_{1}^{\prime} u_{, y}^{2}+2 b_{1}^{\prime} u_{, x} u_{, y}+2 c_{1}^{\prime} u u_{, y}+2 d_{1}^{\prime} v_{, y} u_{, y}+2 e_{1}^{\prime} v_{, x} u_{, y}+2 f_{1}^{\prime} v u_{, y}\right) d x d y \\
& +\iint_{\Gamma_{h}}\left(2 d_{2}^{\prime} v_{, y}^{2}+2 e_{2}^{\prime} v_{, x} v_{, y}+2 f_{2}^{\prime} v v_{, y}+2 a_{2}^{\prime} u_{, y} v_{, y}+2 b_{2}^{\prime} u_{, x} v_{, y}+2 c_{2}^{\prime} u v_{, y}\right) d x d y .
\end{align*}
$$

By the divergence theorem, the left hand side of the above equation can be written as

$$
\int_{\gamma_{h}}\left(\left(u_{, x}^{2}+u_{, y}^{2}\right) y_{\nu}-2\left(u_{, x} u_{, y}\right) x_{\nu}\right)+\left(\left(v_{, x}^{2}+v_{, y}^{2}\right) y_{\nu}-2\left(v_{, x} v_{, y}\right) x_{\nu}\right) d s
$$

where $x_{\nu}, y_{\nu}$ denote the outward normals to the boundary $\gamma_{h}$ (i.e. $\mathrm{AB}+\mathrm{BD}+\mathrm{DC}+\mathrm{CA}$ ). On the initial curve AB the integrand is zero. On CD we have $x_{\nu}=0, y_{\nu}=1, d s=$ $d x$, so the corresponding part of the boundary integral is $E(h)$. On the characteristic edges $B D+C A, x_{\nu}^{2}=y_{\nu}^{2}=\frac{1}{2}$. The corresponding part of the boundary integral can, therefore, be written in the form

$$
\int_{A C+B D} \frac{1}{y_{\nu}}\left[\left(u_{, x} y_{\nu}-u_{, y} x_{\nu}\right)^{2}+\left(v_{, x} y_{\nu}-v_{, y} x_{\nu}\right)^{2}\right] d s
$$

So the equation(B.17) can be written as

$$
\begin{aligned}
& \int_{A C+B D} \frac{1}{y_{\nu}}\left[\left(u_{, x} y_{\nu}-u_{, y} x_{\nu}\right)^{2}+\left(v_{, x} y_{\nu}-v_{, y} x_{\nu}\right)^{2}\right] d s+E(h) \\
= & 2 \iint_{\Gamma_{h}}\left[\left(a_{1} u_{, y}^{2}+b_{1} u_{, x} u_{, y}+c_{1} u u_{, y}\right)+\left(d_{1} u_{, y} v_{, y}+e_{1} u_{, y} v_{, x}+f_{1} u_{, y} v\right)\right] d x d y \\
& +2 \iint_{\Gamma_{h}}\left[\left(d_{2} v_{, y}^{2}+e_{2} v_{, x} v_{, y}+f_{2} v v_{, y}\right)+\left(a_{2} v_{, y} u_{, y}+b_{2} v_{, y} u_{, x}+c_{2} v_{, y} u\right)\right] d x d y \\
= & R,
\end{aligned}
$$

from which we can conclude that

$$
\begin{equation*}
0 \leq E(h) \leq R . \tag{B.19}
\end{equation*}
$$

We estimate the right-hand side of (B.18), observing that $2\left|u_{, x} u_{, y}\right| \leq u_{, x}^{2}+u_{, y}^{2}, 2\left|u u_{, y}\right| \leq$ $u^{2}+u_{, y}^{2}, 2\left|u_{, y} v_{, y}\right| \leq u_{, y}^{2}+v_{, y}^{2}$ etc. Using $M$ to denote an upper bound for the absolute values of the coefficients, we obtain

$$
\begin{align*}
R & \leq \iint_{\Gamma_{h}} M\left(8 u_{, y}^{2}+2 u_{, x}^{2}+2 u^{2}+8 v_{, y}^{2}+2 v_{, x}^{2}+2 v^{2}\right) d x d y  \tag{B.20}\\
& \leq 8 M \iint_{\Gamma_{h}}\left(u_{, x}^{2}+u_{, y}^{2}+u^{2}\right)+\left(v_{, x}^{2}+v_{, y}^{2}+v^{2}\right) d x d y . \tag{B.21}
\end{align*}
$$

With equations (B.12) and (B.13), the above expression arrives at

$$
\begin{align*}
R & \leq 8 M\left(1+h^{2}\right) \iint_{\Gamma_{h}}\left(u_{, x}^{2}+u_{, y}^{2}\right)+\left(v_{, x}^{2}+v_{, y}^{2}\right) d x d y  \tag{B.22}\\
& \leq C \int_{0}^{h} E(y) d y=C \int_{0}^{h} E(\alpha) d \alpha \tag{B.23}
\end{align*}
$$

where $C=8 M\left(1+h^{2}\right)$. If $\forall l$ such that $l>h$, we have

$$
\begin{equation*}
E(h) \leq R \leq C \int_{0}^{h} E(\alpha) d \alpha \leq C \int_{0}^{l} E(\alpha) d \alpha . \tag{B.24}
\end{equation*}
$$

Integrating this relation with respect to $h$ between the limites 0 and $l$, we have

$$
\begin{equation*}
\int_{0}^{l} E(h) d h \leq C l \int_{0}^{l} E(h) d h . \tag{B.25}
\end{equation*}
$$

If $E(h)$ were not zero anywhere in the interval $0 \leq h \leq l$, then it would follow that

$$
\begin{equation*}
1 \leq C l \tag{B.26}
\end{equation*}
$$

which is clearly impossible if we choose $l<1 / C$. Hence in the interval $0<h<l$ we certainly have $E \equiv 0$. Here $l<1 / C$. When $l>1 / C$, it is not necessary to require $E=0$. Repeating the procedure with $y=l, y=2 l, \ldots$ as initial lines, we see after a finite number of steps that $E$ vanishes in the whole triangle $\Gamma$; therefore $u, v$ vanish at the point $p$.

## APPENDIX C <br> Elimination of Dependence on $\mu^{i}$

In the following, we shall remove the terms $\mu^{i}$ and its derivatives, and thus obtain an ordinary differential equation for $\mu^{r}$. To do so, we first introduce $\mu=\mu^{r}+i \mu^{i}$ into the above equation and get

$$
\begin{align*}
& a_{13}^{r} \partial_{x x x x} \mu^{r}-a_{13}^{i} \partial_{x x x x} \mu^{i}+b_{13}^{r} \partial_{x x x} \mu^{r}-b_{13}^{i} \partial_{x x x} \mu^{i}+\cdots+f_{13}^{r}=0  \tag{C.1}\\
& a_{13}^{i} \partial_{x x x x} \mu^{r}+a_{13}^{r} \partial_{x x x x} \mu^{i}+b_{13}^{i} \partial_{x x x} \mu^{r}+b_{13}^{r} \partial_{x x x} \mu^{i}+\cdots+f_{13}^{i}=0 . \tag{C.2}
\end{align*}
$$

We then take $a_{13}^{i} \times(C .2)+a_{13}^{r} \times(C .1)$ to eliminate $\partial_{x x x x} \mu^{i}$ :

$$
\begin{align*}
& a_{14} \partial_{x x x x} \mu^{r}+b_{14} \partial_{x x x} \mu^{r}+c_{14} \partial_{x x} \mu^{r}+d_{14} \partial_{x} \mu^{r}+e_{14} \mu^{r}  \tag{C.3}\\
& \quad+f_{14} \partial_{x x x} \mu^{i}+g_{14} \partial_{x x} \mu^{i}+h_{14} \partial_{x} \mu^{i}+k_{14} \mu^{i}+l_{14}=0 .
\end{align*}
$$

To further eliminate $\partial_{x x x} \mu^{i}$ in the above equation, we shall get an additional equation involving $\partial_{x x x} \mu^{i}$. To get it, we evaluate $f_{14} \times(C .1)+a_{11}^{i} \times \partial_{x}(C .3)$ to obtain

$$
\begin{align*}
& a_{15} \partial_{x^{5}} \mu^{r}+b_{15} \partial_{x x x x} \mu^{r}+c_{15} \partial_{x x x} \mu^{r}+d_{15} \partial_{x x} \mu^{r}+e_{15} \partial_{x} \mu^{r}+f_{15} \mu^{r}  \tag{C.4}\\
&+g_{15} \partial_{x x x} \mu^{i}+h_{15} \partial_{x x} \mu^{i}+k_{15} \partial_{x} \mu^{i}+l_{15} \mu^{i}+m_{15}=0
\end{align*}
$$

and thus we can eliminate $\partial_{x x x} \mu^{i}$ by taking $g_{15} \times(C .3)-f_{14} \times(C .4)$

$$
\begin{align*}
& a_{16} \partial_{x^{5}} \mu^{r}+b_{16} \partial_{x x x x} \mu^{r}+c_{16} \partial_{x x x} \mu^{r}+d_{16} \partial_{x x} \mu^{r}+e_{16} \partial_{x} \mu^{r}+f_{16} \mu^{r}  \tag{C.5}\\
&+g_{16} \partial_{x x} \mu^{i}+h_{16} \partial_{x} \mu^{i}+k_{16} \mu^{i}+l_{16}=0 .
\end{align*}
$$

Proceeding, to eliminiate $\partial_{x x} \mu^{i}$ we get another equation involving $\partial_{x x} \mu^{i}$ by taking $g_{16} \times(C .4)-g_{15} \times \partial_{x}(C .5)$

$$
\begin{align*}
& a_{17} \partial_{x^{6}} \mu^{r}+b_{17} \partial_{x^{5}} \mu^{r}+c_{17} \partial_{x x x x} \mu^{r}+d_{17} \partial_{x x x} \mu^{r}+\cdots+g_{17} \mu^{r}  \tag{C.6}\\
&+h_{17} \partial_{x x} \mu^{i}+k_{17} \partial_{x} \mu^{i}+l_{17} \mu^{i}+m_{17}=0
\end{align*}
$$

and then eliminate $\partial_{x x} \mu^{i}$ by taking $h_{17} \times(C .5)-g_{16} \times(C .6)$ :

$$
\begin{align*}
a_{18} \partial_{x^{6}} \mu^{r}+b_{18} \partial_{x^{5}} \mu^{r}+c_{18} \partial_{x x x x} \mu^{r}+ & d_{18} \partial_{x x x} \mu^{r}+\cdots+g_{18} \mu^{r}  \tag{C.7}\\
& +h_{18} \partial_{x} \mu^{i}+k_{18} \mu^{i}+l_{18}=0 .
\end{align*}
$$

Next, we get another equation involving $\partial_{x} \mu^{i}$ by taking $h_{18} \times(C .6)-h_{17} \times \partial_{x}(C .7)$

$$
\begin{align*}
a_{19} \partial_{x^{7}} \mu^{r}+b_{19} \partial_{x^{6}} \mu^{r}+c_{19} \partial_{x^{5}} \mu^{r}+ & d_{19} \partial_{x x x x} \mu^{r}+\cdots+h_{19} \mu^{r}  \tag{C.8}\\
& +k_{19} \partial_{x} \mu^{i}+l_{19} \mu^{i}+m_{19}=0,
\end{align*}
$$

and eliminate $\partial_{x} \mu^{i}$ by taking $k_{19} \times(C .7)-h_{18} \times(C .8)$ :

$$
\begin{align*}
a_{20} \partial_{x^{7}} \mu^{r}+b_{20} \partial_{x^{6}} \mu^{r}+c_{20} \partial_{x^{5}} \mu^{r}+d_{20} \partial_{x x x x} \mu^{r} & +\cdots+h_{20} \mu^{r}  \tag{C.9}\\
& +k_{20} \mu^{i}+l_{20}=0 .
\end{align*}
$$

We then get another equation involving $\mu^{i}$ by taking $k_{20} \times(C .8)-k_{19} \times \partial_{x}(C .9)$ :

$$
\begin{align*}
a_{21} \partial_{x^{8}} \mu^{r}+b_{21} \partial_{x^{7}} \mu^{r}+c_{21} \partial_{x^{6}} \mu^{r}+d_{21} \partial_{x^{5}} \mu^{r} & +\cdots+k_{21} \mu^{r}  \tag{C.10}\\
& +l_{21} \mu^{i}+m_{21}=0,
\end{align*}
$$

and finally eliminate $\mu^{i}$ to obtain an eighth order ordinary differential equation for $\mu^{r}$ by taking $l_{21} \times(C .9)-k_{20} \times(C .10)$ :

$$
\begin{equation*}
a_{22} \partial_{x^{8}} \mu^{r}+b_{22} \partial_{x^{7}} \mu^{r}+c_{22} \partial_{x^{6}} \mu^{r}+d_{22} \partial_{x^{5}} \mu^{r}+\cdots+k_{22} \mu^{r}+l_{22}=0 . \tag{C.11}
\end{equation*}
$$


[^0]:    Portions of this chapter previously appeared as: Y. Zhang, A. A. Oberai, P. E. Barbone, and I. Harari, "Solution of the time-harmonic viscoelastic inverse problem with interior data in two dimensions," Int. J. Numer. Meth. Eng., doi: 10.1002/nme.4372, 2012.

