Residual-based stabilized formulation for the solution of inverse elliptic PDE

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Abstract

In several fields of science and engineering, such as thermometry, elasticity imaging, and geophysics, the solution to an inverse problem with interior data is sought, wherein the forward model is in the form of an elliptic partial differential equation. A common approach to solving these problems is to pose them as a constrained minimization problem, where the difference between the measured and a predicted response is minimized under the constraint that the predicted response satisfy the forward elliptic model. The optimization parameters represent the spatial distribution of the material properties in the forward model, and the data mismatch is measured in the L_2 norm. In this manuscript we consider an instantiation of this problem, where the forward problem is that of linear plane stress elasticity, or equivalently that of linear heat/hydraulic conduction. We demonstrate that the linearized version of the saddle point problem obtained from the minimization problem inherits some stability from the forward elliptic problem. In particular, it is stable for the response variable and the Lagrange multiplier, but not for the material property field. This lack of stability implies that we are unable to prove optimal convergence with mesh refinement for the overall problem. We overcome this difficulty by adding to the saddle point problem a residual-based term that provides sufficient stability, and prove optimal convergence in an energy-like norm. We verify these estimates through simple numerical examples. We note that while we have considered a specific model for an inverse elliptic problem in this manuscript, similar ideas could be developed for a broad class of inverse elliptic problems.

Key words: Inverse elliptic problem, constrained minimization problem, saddle point problem, residual-based stabilization formulation

1. Introduction

The problem of determining the material parameters from a given set of interior measurements is an inverse problem. The determination of shear modulus from displacement measurements [1], thermal conductivity from temperature measurements [2], and the aquifer permeability from water pressure head measurements [3], are examples of these types of problems.

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In the inverse two-dimensional plane stress problem, given the measured displacement components $\tilde{\boldsymbol{u}} = [\tilde{u}_1, \tilde{u}_2]$, the aim is to determine the shear modulus distribution, μ, such that,

$$\nabla \cdot (\mu \mathbb{C} \nabla \tilde{\boldsymbol{u}}) = \mathbf{0}, \tag{1}$$

where $\mathbb{C}$ is a 4$^{\text{th}}$ order tensor given by $\mathbb{C}_{ijkl} = \delta_{ij}\delta_{kl} + \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})$.

The appropriate boundary condition for the above system to be well posed is that μ is prescribed at a point of the domain, or the average value of μ is prescribed. That is,

$$\frac{\int_\Omega \mu \, d\Omega}{\int_\Omega d\Omega} = \mu_0. \tag{2}$$

We note that the system (1), along with the condition (2) can represent other inverse problems, such as the inverse heat conduction problem, and the inverse hydraulic problem. In the two-dimensional heat conduction (hydraulic) problem $\tilde{u}_1$ and $\tilde{u}_2$ represent two independent temperature (pressure) field measurements, μ represents the thermal (hydraulic) conductivity, and the tensor $\mathbb{C}$ is given by $\mathbb{C}_{ijkl} = \delta_{ijkl}$.

Several methods have been proposed to solve the inverse problem defined in (1) and (2). They can be broadly classified as “direct” or minimization methods. In direct methods, the measured displacement field is directly inserted in (1), and this equation is interpreted as an equation for the shear modulus μ. This system yields two hyperbolic PDEs for μ that must be solved simultaneously with no boundary data. The simplest approach to solve these equations is to consider a least-squares formulation [4]. However, it has been shown that this results in numerical results that are overly dissipative, especially for rough measured data [2]. An alternate is to consider the adjoint-weighted variational equation, where the original equation for μ is weighted by its adjoint operating on the weighting functions [5, 6]. This is in contrast to the least squares approach, where the original equation is weighted by the same operator acting on the weighting function. In results presented in [2] the adjoint-weighted equations have demonstrated superior performance (particularly for sharp spatial changes) when compared with the least-squares method.

Direct methods are appealing in that they are easily implemented and computationally inexpensive. On the other hand, since they use the measured displacement data and its derivatives directly in the equation of equilibrium, they have difficulty in handling noisy data obtained from real experiments. Recently, methods that smooth this data under minimal assumptions on the underlying modulus distributions and address this issue to some extent have been proposed [7].

Another class of methods of solving the inverse problem circumvent this problem entirely. In these methods the displacement field that appears in (1) is a smoother, predicted displacement field, which is computed by solving this equation for a given distribution of μ. The difference between the predicted and measured displacement field is minimized by varying the spatial distribution of the shear modulus, and the shear modulus is smoothed through the use of a regularization term. This approach leads to a regularized constrained minimization problem, where the difference between the predicted and measured displacement fields is minimized under the constraint of the equation of equilibrium. When the difference between the predicted and measured fields is measured in the L_2 norm, we are lead to the problem considered in [8, 9]. Another choice is to measure the difference between the predicted and measured displacements in the “energy” norm induced by the shear modulus distribution as described in [10, 11].

The solution of the constrained minimization problem leads to variational equations that are most naturally approximated using the finite element method. A question then arises about the convergence of the finite dimensional solution to the exact solution. In [3], for the L_2 minimization approach, in the context of the inverse hydraulic problem, with a single measurement it was shown that for perfect data, the hydraulic conductivity converges sub-optimally in the L_2 norm to the exact modulus distribution. Further, in [10, 12], it was shown that the finite element discretization of the variational formulation associated with a mismatch measured in the energy norm, converges at optimal rates with the mesh refinement.

In this paper, following [3], we consider the L_2 minimization problem. However we consider the two-dimensional plane stress problem, or equivalently the heat conduction problem with two measurements. Our Lagrangian is comprised of an L_2 displacement mismatch term, and a constraint equation. We observe that the linearized version of the variational equations obtained from the Lagrangian lacks stability. We address this problem by adding to the variational equations terms that are driven by the residual of the constraint. We note that these additional terms add stability to the linearized version of the variational equation. This approach is similar to an augmented Lagrangian method [12, 13, 14, 15, 16]. In the augmented Lagrangian approach, a penalty term that is driven by the constraint is added to the Lagrangian. In contrast to this, we append a term driven by the constraint directly to the variational equation derived from the stationarity of the original Lagrangian. This allows us to consider a large range of the possible terms in the variational equations than the augmented Lagrangian method, which only allows for least squares type terms.

The layout of the remainder of this manuscript is as follows. In Section 2, we pose the inverse problem as L_2 constrained minimization problem. We construct the Lagrangian to incorporate the constraint using Lagrange multipliers, and then derive the non-linear saddle point problem (SPP) by setting its first variation to zero. We discretize the non-linear SPP using standard finite element shape functions. We show that the linearized version of the saddle point problem lacks stability. We account for this by adding a residual based stabilization term to the saddle point problem. In Section 3, we consider the convergence of the stabilized formulation with mesh refinement and prove optimal convergence rates in an "energy norm". In Section 4, we perform numerical verification of the stabilized formulation, and demonstrate the optimal convergence rates using two sample problems.

2. Problem Formulation

We pose the inverse problem as a constrained minimization problem, where one seeks a sufficiently smooth field $\boldsymbol{u}$, such that the distance between the measured field $\tilde{\boldsymbol{u}}$, and $\boldsymbol{u}$ is minimized in the least-square sense, while the shear modulus μ, and the predicted field $\boldsymbol{u}$ satisfy weak form of the governing partial differential equation (1)-(2).

The predicted displacement field belongs to the space of trial functions,

$$\mathcal{U} \equiv \{\boldsymbol{u} | \boldsymbol{u} \in H^1(\Omega), \boldsymbol{u} = \tilde{\boldsymbol{u}} \text{ on } \partial\Omega\}, \tag{3}$$

and the shear modulus belongs to the space $\mathcal{M}$ defined as,

$$\mathcal{M} \equiv \{\mu | \mu \in H^1(\Omega), \frac{\int_\Omega \mu \, d\Omega}{\int_\Omega \, d\Omega} = \mu_0\}. \tag{4}$$

We can state the minimization problem as follows,

$$\left.\begin{aligned} \mu* = \underset{\mu \in \mathcal{M}}{\text{argmin}}\ \tfrac{1}{2}\int_\Omega |\boldsymbol{u}-\tilde{\boldsymbol{u}}|^2\ d\Omega, \\ s.t,\ a(\boldsymbol{w},\boldsymbol{u};\mu) = 0,\ \forall\ \boldsymbol{w} \in \mathcal{W}. \end{aligned}\right\} \tag{5}$$

The weighting function space $\mathcal{W}$, and the tri-linear form $a(\cdot,\cdot;\cdot)$ are defined as,

$$\mathcal{W} \equiv \{\boldsymbol{w}|\boldsymbol{w} \in H^1(\Omega), \boldsymbol{w} = 0 \text{ on } \partial\Omega\}, \tag{6}$$

$$a(\boldsymbol{w},\boldsymbol{u};\mu) \equiv \int_\Omega \mu(\nabla\boldsymbol{w} : \mathbb{C}\nabla\boldsymbol{u})\ d\Omega. \tag{7}$$

To obtain a saddle point system from the constrained minimization problem (5), we construct the following Lagrangian, where the given constraint is imposed through the Lagrange multipliers $\boldsymbol{\lambda} \in \mathcal{W}$,

$$\mathcal{L}(\boldsymbol{\lambda},\boldsymbol{u};\mu) \equiv \frac{1}{2}\int_\Omega |\boldsymbol{u}-\tilde{\boldsymbol{u}}|^2\ d\Omega + a(\boldsymbol{\lambda},\boldsymbol{u};\mu). \tag{8}$$

To attain the minimum, the Lagrangian needs to satisfy the first order optimality condition (the so-called Karush Kuhn Tucker (KKT) conditions [17]), that is $\delta\mathcal{L}(\boldsymbol{\lambda},\boldsymbol{u};\mu) = 0$. Setting the first variations of the Lagrangian with respect to $\{\boldsymbol{u},\mu,\boldsymbol{\lambda}\}$ to zero, we arrive at the following Euler-Lagrange equations,

$$D_{\boldsymbol{u}}\mathcal{L}\ \cdot \boldsymbol{v} + D_\mu\mathcal{L}\ \cdot q + D_\lambda\mathcal{L}\ \cdot \boldsymbol{w} = 0,\ \forall\ \{\boldsymbol{v},q,\boldsymbol{w}\} \in \mathcal{W}\times\mathcal{Q}\times\mathcal{W}. \tag{9}$$

Using the definition of the Lagrangian, this yields, find $\{\boldsymbol{u},\mu,\boldsymbol{\lambda}\} \in \mathcal{U}\times\mathcal{M}\times\mathcal{U}$, such that,

$$(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{v}) + a(\boldsymbol{\lambda},\boldsymbol{v};\mu) + a(\boldsymbol{\lambda},\boldsymbol{u};q) + a(\boldsymbol{w},\boldsymbol{u};\mu) = 0, \forall\{\boldsymbol{v},q,\boldsymbol{w}\} \in \mathcal{W}\times\mathcal{Q}\times\mathcal{W}. \tag{10}$$

The first term in the equation above originates from the data mismatch term and the other three terms originate from the constraint equation. Further, the space $\mathcal{Q}$ is defined as follows,

$$\mathcal{Q} \equiv \{q|q \in H^1(\Omega),\ \int_\Omega q\ d\Omega = 0\}. \tag{11}$$

When the variational form (10) is approximated using the standard Galerkin method, we are led to the following finite dimensional problem.

Find $\{\boldsymbol{u}^h,\mu^h,\boldsymbol{\lambda}^h\} \in \mathcal{U}^h\times\mathcal{M}^h\times\mathcal{U}^h$, such that,

$$\begin{aligned}(\boldsymbol{u}^h-\tilde{\boldsymbol{u}}^h,\boldsymbol{v}^h) + a(\boldsymbol{\lambda}^h,\boldsymbol{v}^h;\mu^h) + a(\boldsymbol{\lambda}^h,\boldsymbol{u}^h;q^h) + a(\boldsymbol{w}^h,\boldsymbol{u}^h;\mu^h) = 0,& \\ \forall\{\boldsymbol{v}^h,q^h,\boldsymbol{w}^h\} \in \mathcal{W}^h\times\mathcal{Q}^h\times\mathcal{W}^h.&\end{aligned} \tag{12}$$

Equation (12) is a non-linear equation, and which may be solved using Newton's method. As a result we obtain the obtain the following linear saddle point problem for the increment at each Newton step.

Given $\{\boldsymbol{u}^h,\mu^h,\boldsymbol{\lambda}^h\} \in \mathcal{U}^h\times\mathcal{M}^h\times\mathcal{U}^h$, find $\{\delta\boldsymbol{u}^h,\delta\mu^h,\delta\boldsymbol{\lambda}^h\} \in \mathcal{W}^h\times\mathcal{Q}^h\times\mathcal{W}^h$, such that,$\forall\{\boldsymbol{v}^h,q^h,\boldsymbol{w}^h\} \in \mathcal{W}^h\times\mathcal{Q}^h\times\mathcal{W}^h$, the following holds,

$$(\delta\boldsymbol{u}^h,\boldsymbol{v}^h) \qquad +a(\boldsymbol{\lambda}^h,\boldsymbol{v}^h;\delta\mu^h) \qquad +a(\delta\boldsymbol{\lambda}^h,\boldsymbol{v}^h;\mu^h) \qquad = -(\boldsymbol{u}^h-\tilde{\boldsymbol{u}}^h,\boldsymbol{v}^h)$$

$$
\begin{aligned}
&&&&- a(\boldsymbol{\lambda}^h, \boldsymbol{v}^h; \mu^h), \quad (14)\\
a(\boldsymbol{\lambda}^h, \delta\boldsymbol{u}^h; q^h) &\quad +0 &&\quad +a(\delta\boldsymbol{\lambda}^h, \boldsymbol{u}^h; q^h) &= - a(\boldsymbol{\lambda}^h, \boldsymbol{u}^h; q^h), \quad (15)\\
a(\boldsymbol{w}^h, \delta\boldsymbol{u}^h; \mu^h) &\quad +a(\boldsymbol{w}^h, \boldsymbol{u}^h; \delta\mu^h) &&\quad +0 &= - a(\boldsymbol{w}^h, \boldsymbol{u}^h; \mu^h). \quad (16)
\end{aligned}
$$

To examine the well-posedness of the above system, we consider the inf-sup stability of the whole system [18]. For convenience, we rewrite (14) to (16) as,

$$
A(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}, \{\delta\boldsymbol{u}^h, \delta\mu^h, \delta\boldsymbol{\lambda}^h\}) = L(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}), \tag{17}
$$

where,

$$
\begin{aligned}
A(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}, \{\delta\boldsymbol{u}^h, \delta\mu^h, \delta\boldsymbol{\lambda}^h\}) \equiv\ & (\delta\boldsymbol{u}^h, \boldsymbol{v}^h) + a(\boldsymbol{\lambda}^h, \boldsymbol{v}^h; \delta\mu^h) + a(\boldsymbol{\lambda}^h, \delta\boldsymbol{u}^h; q^h)\\
& + a(\delta\boldsymbol{\lambda}^h, \boldsymbol{v}^h; \mu^h) + a(\boldsymbol{w}^h, \delta\boldsymbol{u}^h; \mu^h)\\
& + a(\delta\boldsymbol{\lambda}^h, \boldsymbol{u}^h; q^h) + a(\boldsymbol{w}^h, \boldsymbol{u}^h; \delta\mu^h),
\end{aligned} \tag{18}
$$

$$
L(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}) \equiv - (\boldsymbol{u}^h - \tilde{\boldsymbol{u}}^h, \boldsymbol{v}^h) - a(\boldsymbol{\lambda}^h, \boldsymbol{v}^h; \mu^h) - a(\boldsymbol{\lambda}^h, \boldsymbol{u}^h; q^h) - a(\boldsymbol{w}^h, \boldsymbol{u}^h; \mu^h). \tag{19}
$$

When evaluating the inf-sup condition for a bilinear form, for every function in the weighting function space one is allowed to make the best choice (the sup-part of the condition) that maximizes the stability parameter. Recognizing this, for every $\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}$ in the weighting function slot, we consider $\{\boldsymbol{w}^h, q^h, \boldsymbol{v}^h\}$ in the trial function slot, (note that the slots for $\boldsymbol{w}^h$ and $\boldsymbol{v}^h$ in the test function are switched) to get,

$$
\begin{aligned}
A(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}, \{\boldsymbol{w}^h, q^h, \boldsymbol{v}^h\}) =& (\boldsymbol{v}^h, \boldsymbol{w}^h) + a(\boldsymbol{\lambda}^h, \boldsymbol{w}^h + \boldsymbol{v}^h; q^h) + a(\boldsymbol{w}^h, \boldsymbol{w}^h; \mu^h)\\
& + a(\boldsymbol{v}^h, \boldsymbol{v}^h; \mu^h) + a(\boldsymbol{w}^h + \boldsymbol{v}^h, \boldsymbol{u}^h; q^h).
\end{aligned} \tag{20}
$$

From the definition of $a(\cdot, \cdot; \cdot)$ (equation 7), we conclude that the two terms, $a(\boldsymbol{v}^h, \boldsymbol{v}^h; \mu^h)$ and $a(\boldsymbol{w}^h, \boldsymbol{w}^h; \mu^h)$, on the right hand side of (20) indicate that we have coercivity in the gradients of $\boldsymbol{v}^h$ and $\boldsymbol{w}^h$. However, there is no positive term for q^h. Thus if we were to propose a numerical method based on (17) using standard finite element shape functions, we would not be able to demonstrate its inf-sup stability.

This is rectified by adding a residual-based stabilization term to the discrete variational equation (12). The stabilized variational formulation is given by:

Find $\{\boldsymbol{u}^h, \mu^h, \boldsymbol{\lambda}^h\} \in \mathcal{U}^h \times \mathcal{M}^h \times \mathcal{W}^h$, such that,

$$
\begin{aligned}
(\boldsymbol{u}^h - \tilde{\boldsymbol{u}}^h, \boldsymbol{v}^h) + a(\boldsymbol{\lambda}^h, \boldsymbol{v}^h; \mu^h) + a(\boldsymbol{\lambda}^h, \boldsymbol{u}^h; q^h) + a(\boldsymbol{w}^h, \boldsymbol{u}^h; \mu^h) + \tau_s(\mathbb{C}\nabla\boldsymbol{u}^h \cdot \nabla q^h, \nabla \cdot (\mu^h \mathbb{C}\nabla\boldsymbol{u}^h))_{\tilde{\Omega}} = 0,&\\
\forall\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\} \in \mathcal{W}^h \times \mathcal{Q}^h \times \mathcal{W}^h,&
\end{aligned} \tag{21}
$$

where, τ_s is the stabilization parameter, $\tilde{\Omega}$ is the union of the interior of each of the finite element domains. The form of the additional term is motivated by Streamwise Upwind Petrov Galerkin (SUPG) method, [19, 20], and the adjoint-weighted equation [21, 2]. We apply Newton's method to this stabilized formulation and arrive at the following linear problem for the increments.

Given $\{\boldsymbol{u}^h, \mu^h, \boldsymbol{\lambda}^h\} \in \mathcal{U}^h \times \mathcal{M}^h \times \mathcal{W}^h$, find $\{\delta\boldsymbol{u}^h, \delta\mu^h, \delta\boldsymbol{\lambda}^h\} \in \mathcal{W}^h \times \mathcal{Q}^h \times \mathcal{W}^h$, such that,

$$
A_s(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}, \{\delta\boldsymbol{u}^h, \delta\mu^h, \delta\boldsymbol{\lambda}^h\}) = L_s(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}), \forall\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\} \in \mathcal{W}^h \times \mathcal{Q}^h \times \mathcal{W}^h. \tag{22}
$$

where,

$$
\begin{aligned}
A_s(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}, \{\delta\boldsymbol{u}^h, \delta\mu^h, \delta\boldsymbol{\lambda}^h\}) \equiv & A(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}, \{\delta\boldsymbol{u}^h, \delta\mu^h, \delta\boldsymbol{\lambda}^h\}) \\
& + \tau_s(\mathbb{C}\nabla\delta\boldsymbol{u}^h \cdot \nabla q^h, \nabla\cdot(\mu^h\mathbb{C}\nabla\boldsymbol{u}^h))_{\tilde{\Omega}} \\
& + \tau_s(\mathbb{C}\nabla\boldsymbol{u}^h \cdot \nabla q^h, \nabla\cdot(\delta\mu^h\mathbb{C}\nabla\boldsymbol{u}^h))_{\tilde{\Omega}} \\
& + \tau_s(\mathbb{C}\nabla\boldsymbol{u}^h \cdot \nabla q^h, \nabla\mu^h \cdot \mathbb{C}\nabla\delta\boldsymbol{u}^h)_{\tilde{\Omega}},
\end{aligned} \tag{23}
$$

$$
L_s(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}) \equiv L(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}) + \tau_s(\mathbb{C}\nabla\boldsymbol{u}^h \cdot \nabla q^h, \nabla\cdot(\mu^h\mathbb{C}\nabla\boldsymbol{u}^h))_{\tilde{\Omega}}. \tag{24}
$$

We note that we have dropped a term that contains the second order derivative of $\delta\boldsymbol{u}^h$ from the consistent linearization of the stabilized problem in (23). Excluding this term simplifies the analysis of the stability of this formulation. It does however imply that we may not attain quadratic convergence for the proposed Newton method.

In the preparation of the convergence analysis, we introduce the continuous problem as, find $\{\delta\boldsymbol{u}, \delta\mu, \delta\boldsymbol{\lambda}\} \in \mathcal{W} \times \mathcal{Q} \times \mathcal{W}$, such that,

$$
A_s(\{\boldsymbol{v}, q, \boldsymbol{w}\}, \{\delta\boldsymbol{u}, \delta\mu, \delta\boldsymbol{\lambda}\}) = L_s(\{\boldsymbol{v}, q, \boldsymbol{w}\}),\ \forall\ \ \{\boldsymbol{v}, q, \boldsymbol{w}\} \in \mathcal{W} \times \mathcal{Q} \times \mathcal{W}. \tag{25}
$$

In order for the continuous problem to have a well defined solution, we assume that the functions $\mu^h, \boldsymbol{u}^h, \boldsymbol{\lambda}^h$, that appear as spatially-varying parameters in the above equation have sufficient regularity.

The main result of this manuscript is a proof of the convergence of the solution of the linear stabilized saddle point problem (22) to its continuous counterpart (25). This is accomplished in the following section with four lemmas and one theorem. In Lemma 3.1, we establish the "orthogonality" of the error with respect to the discrete weighting function space. In Lemma 3.2, we establish inf-sup stability for the stabilized problem through a modified Garding-like inequality. Lemma 3.3 is a statement of the continuity of the tri-linear form A_s. In Lemma 3.5, we employ the Aubin-Nitsche duality argument to bound the L_2 error by errors in high-order norms. Finally in Theorem 3.6, we prove the final convergence result.

3. Convergence of the Linearized Stabilized Problem

First we define some notation utilized in the remainder of this manuscript:

- Let $\boldsymbol{v}$ denote a vector, $\mathbf{A}$ denote a 2^{nd} order tensor, then,

$$
\mid \boldsymbol{v} \mid = (\sum v_i^2)^{\frac{1}{2}}, \qquad \text{(2-norm of a vector)} \tag{26}
$$

$$
\mid \mathbf{A} \mid = \sup_{|\boldsymbol{v}|\neq 0} \frac{\mid \mathbf{A}\boldsymbol{v} \mid}{\mid \boldsymbol{v} \mid}, \qquad \text{(operator induced 2-norm of tensor)} \tag{27}
$$

$$
\mid \mathbf{A} \mid_F = (\mathbf{A} : \mathbf{A})^{\frac{1}{2}}. \qquad \text{(Frobenius norm of a tensor)} \tag{28}
$$

- From equivalence of finite-dimensional norms, $\exists\ c_1$, c_2, both positive and finite, *s.t*,

$$
c_1 \mid \mathbf{A} \mid_F \ \leq \ \mid \mathbf{A} \mid \ \leq c_2 \mid \mathbf{A} \mid_F . \tag{29}
$$

- We define a norm of a 4^{th} order tensor $\mathbb{D}$, as follows,

$$
\mid \mathbb{D} \mid_s = \sup_{|\mathbf{A}|_F \neq 0} \frac{\mid \mathbb{D}\mathbf{A} \mid_F}{\mid \mathbf{A} \mid_F}, \tag{30}
$$

  where $\mathbf{A}$ denotes a symmetric second-order tensor. It is easy to verify that for the tensor in (1), $\mid \mathbb{C} \mid_s = 2$.

- To represent the L_2, and L_∞ norms of a scalar, a vector, or a tensor valued function, we use $\|\cdot\|$, and $\|\cdot\|_\infty$, respectively.

We are now ready to analyze the linearized saddle point problem.

Lemma 3.1. *Let, $\{\delta\boldsymbol{u}, \delta\mu, \delta\boldsymbol{\lambda}\}$ and $\{\delta\boldsymbol{u}^h, \delta\mu^h, \delta\boldsymbol{\lambda}^h\}$ be given by (25) and (22) respectively, and let $\boldsymbol{e_u} = \delta\boldsymbol{u} - \delta\boldsymbol{u}^h \in \mathcal{W}$, $e_\mu = \delta\mu - \delta\mu^h \in \mathcal{Q}$, and $\boldsymbol{e_\lambda} = \delta\boldsymbol{\lambda} - \delta\boldsymbol{\lambda}^h \in \mathcal{W}$, represent the error in the finite element approximation, then for any $\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\} \in \mathcal{W}^h \times \mathcal{Q}^h \times \mathcal{W}^h$, we have,*

$$
A_s(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}, \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e_\lambda}\}) = 0, \ \forall \{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\} \in \mathcal{W}^h \times \mathcal{Q}^h \times \mathcal{W}^h. \tag{31}
$$

Proof. Since $\mathcal{W}^h \subset \mathcal{W}$, and $\mathcal{Q}^h \subset \mathcal{Q}$, from (25), we have,

$$
A_s(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}, \{\delta\boldsymbol{u}, \delta\mu, \delta\boldsymbol{\lambda}\}) = L_s(\{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\}), \ \ \forall \{\boldsymbol{v}^h, q^h, \boldsymbol{w}^h\} \in \mathcal{W}^h \times \mathcal{Q}^h \times \mathcal{W}^h. \tag{32}
$$

Subtracting (22) from (32), we have the desired result. □

Equation (31) states that the error $\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e_\lambda}\}$ is "orthogonal" to the finite-dimensional weighting function spaces.

Our next lemma is a statement about the stability of the problem. Like the Garding's inequality [22], we need to add L_2 terms to the variational formulation. However, in addition to this we also have to switch the slots for the displacements and Lagrange multipliers in the trial solution in order to prove stability. For this reason we refer to this as a modified Garding's inequality.

Lemma 3.2. (Modified Garding's Inequality) *[22]Let*

1. *$\boldsymbol{u}$ be such that there exist finite positive constants γ_0, γ_∞, where, $0 < \gamma_0 < \gamma_\infty < \infty$, s.t,*

$$
\gamma_0 \|\nabla q\| \leq \|\mathbb{C}\nabla^s \boldsymbol{u} \cdot \nabla q\| \leq \gamma_\infty \|\nabla q\|, \tag{33}
$$

2. *μ be such that $\inf(\mu) \geq \mu_{min} > 0$,*
3. *τ_s be such that,*

$$
\tau_s \leq \frac{\gamma_0^2 \mu_{min}}{2c_2^2 \mid \mathbb{C} \mid_s^2 \left(\|\nabla \cdot (\mu \mathbb{C} \nabla^s \boldsymbol{u})\|_\infty^2 + \gamma_\infty^2 \|\nabla \mu\|_\infty^2 \right)}, \tag{34}
$$

   where c_2, and $\mid \mathbb{C} \mid_s$ are defined in (29), and (30), respectively.

Then for any $\{\boldsymbol{v}, q, \boldsymbol{w}\} \in \mathcal{W} \times \mathcal{Q} \times \mathcal{W}$,

$$
A_s(\{\boldsymbol{v}, q, \boldsymbol{w}\}, \{\boldsymbol{w}, q, \boldsymbol{v}\}) + K_1 \|\boldsymbol{v}\|^2 + K_2 \|q\|^2 + K_3 \|\boldsymbol{w}\|^2 \geq \frac{1}{4} \|\!|\!| \{\boldsymbol{v}, q, \boldsymbol{w}\} |\!|\!\|^2,
$$

where,

$$
\|\!|\!| \{\boldsymbol{v}, q, \boldsymbol{w}\} |\!|\!\|^2 \equiv \mu_{min}(\|\nabla^s \boldsymbol{v}\|^2 + \|\nabla^s \boldsymbol{w}\|^2) + \tau_s \gamma_0^2 \|\nabla q\|^2, \tag{35}
$$

$$K_1 = 1, \tag{36}$$

$$K_2 = \frac{8 \ \mid \mathbb{C} \mid_s^2 (||\nabla^s \boldsymbol{\lambda}||_\infty^2 + ||\nabla^s \boldsymbol{u}||_\infty^2)}{3 \ \mu_{min}} + \frac{\tau_s \gamma_\infty^2 ||\nabla \cdot \mathbb{C} \nabla^s \boldsymbol{u}||_\infty^2}{\gamma_0^2}, \tag{37}$$

$$K_3 = \frac{1}{4}. \tag{38}$$

Proof. For any $\{\boldsymbol{v}, q, \boldsymbol{w}\} \in \mathcal{W} \times Q \times \mathcal{W}$, we have the following from the definition of A_s,

$$\begin{aligned} A_s(\{\boldsymbol{v}, q, \boldsymbol{w}\}, \{\boldsymbol{w}, q, \boldsymbol{v}\}) \geq &- |(\boldsymbol{v}, \boldsymbol{w})| - |a(\boldsymbol{\lambda}, \boldsymbol{w}; q)| + a(\boldsymbol{w}, \boldsymbol{w}; \mu) - |a(\boldsymbol{\lambda}, \boldsymbol{v}; q)| - |a(\boldsymbol{w}, \boldsymbol{u}; q)| \\ &+ a(\boldsymbol{v}, \boldsymbol{v}; \mu) - |a(\boldsymbol{v}, \boldsymbol{u}; q)| - \tau_s |(\mathbb{C} \nabla \boldsymbol{w} \cdot \nabla q, \nabla \cdot (\mu \mathbb{C} \nabla \boldsymbol{u}))| \\ &+ \tau_s (\mathbb{C} \nabla \boldsymbol{u} \cdot \nabla q, \nabla q \cdot \mathbb{C} \nabla \boldsymbol{u}) - \tau_s |(\mathbb{C} \nabla \boldsymbol{u} \cdot \nabla q, q(\nabla \cdot \mathbb{C} \nabla \boldsymbol{u}))| \\ &- \tau_s |(\mathbb{C} \nabla \boldsymbol{u} \cdot \nabla q, \nabla \mu \cdot \mathbb{C} \nabla \boldsymbol{w})|. \end{aligned} \tag{39}$$

We now use Young's inequality in every negative term on the right hand side, the definition of the norm induced by a 4^{th} order tensor (30) in terms 2,4,5,7,8 and 11, and the equivalence of norms (29) in terms 8 and 11, to arrive at,

$$\begin{aligned} A_s(\{\boldsymbol{v}, q, \boldsymbol{w}\}, \{\boldsymbol{w}, q, \boldsymbol{v}\}) \geq &- \frac{1}{2}\left(\frac{||\boldsymbol{v}||^2}{\epsilon_1} + \epsilon_1 ||\boldsymbol{w}||^2\right) - \frac{\mid \mathbb{C} \mid_s ||\nabla^s \boldsymbol{\lambda}||_\infty}{2}\left(\frac{||q||^2}{\epsilon_2} + \epsilon_2 ||\nabla^s \boldsymbol{w}||^2\right) \\ &+ \mu_{min} ||\nabla^s \boldsymbol{w}||^2 - \frac{\mid \mathbb{C} \mid_s ||\nabla^s \boldsymbol{\lambda}||_\infty}{2}\left(\frac{||q||^2}{\epsilon_3} + \epsilon_3 ||\nabla^s \boldsymbol{v}||^2\right) \\ &- \frac{\mid \mathbb{C} \mid_s ||\nabla^s \boldsymbol{u}||_\infty}{2}\left(\frac{||q||^2}{\epsilon_4} + \epsilon_4 ||\nabla^s \boldsymbol{w}||^2\right) \\ &+ \mu_{min} ||\nabla^s \boldsymbol{v}||^2 - \frac{\mid \mathbb{C} \mid_s ||\nabla^s \boldsymbol{u}||_\infty}{2}\left(\frac{||q||^2}{\epsilon_5} + \epsilon_5 ||\nabla^s \boldsymbol{v}||^2\right) \\ &- \frac{\tau_s c_2 \mid \mathbb{C} \mid_s ||\nabla \cdot (\mu \mathbb{C} \nabla^s \boldsymbol{u})||_\infty}{2}\left(\frac{||\nabla q||^2}{\epsilon_6} + \epsilon_6 ||\nabla^s \boldsymbol{w}||^2\right) \\ &+ \tau_s ||\mathbb{C} \nabla^s \boldsymbol{u} \cdot \nabla q||^2 - \frac{\tau_s ||\nabla \cdot \mathbb{C} \nabla^s \boldsymbol{u}||_\infty}{2}\left(\frac{||\mathbb{C} \nabla^s \boldsymbol{u} \cdot \nabla q||^2}{\epsilon_7} + \epsilon_7 ||q||^2\right) \\ &- \frac{\tau_s c_2 \mid \mathbb{C} \mid_s ||\nabla \mu||_\infty}{2}\left(\frac{||\mathbb{C} \nabla^s \boldsymbol{u} \cdot \nabla q||^2}{\epsilon_8} + \epsilon_8 ||\nabla^s \boldsymbol{w}||^2\right). \end{aligned} \tag{40}$$

We make the following choices for ϵ_i,

$$\begin{aligned} &\epsilon_1 = \frac{1}{2}, \ \epsilon_2 = \frac{\mu_{min}}{4 \mid \mathbb{C} \mid_s ||\nabla^s \boldsymbol{\lambda}||_\infty}, \ \epsilon_3 = 3\epsilon_2, \ \epsilon_4 = \frac{\mu_{min}}{4 \mid \mathbb{C} \mid_s ||\nabla^s \boldsymbol{u}||_\infty}, \ \epsilon_5 = 3\epsilon_4, \\ &\epsilon_6 = \frac{2c_2 \mid \mathbb{C} \mid_s ||\nabla \cdot (\mu \mathbb{C} \nabla^s \boldsymbol{u})||_\infty}{\gamma_0^2}, \ \epsilon_7 = \frac{2\gamma_\infty^2 ||\nabla \cdot \mathbb{C} \nabla^s \boldsymbol{u}||_\infty}{\gamma_0^2}, \\ &\epsilon_8 = \frac{2\gamma_\infty^2 c_2 \mid \mathbb{C} \mid_s ||\nabla \mu||_\infty}{\gamma_0^2}, \end{aligned}$$

to arrive at,

$$A_s(\{\boldsymbol{v}, q, \boldsymbol{w}\}, \{\boldsymbol{w}, q, \boldsymbol{v}\}) \geq - ||\boldsymbol{v}||^2 - \left(\frac{8 \ \mid \mathbb{C} \mid_s^2 (||\nabla^s \boldsymbol{\lambda}||_\infty^2 + ||\nabla^s \boldsymbol{u}||_\infty^2)}{3 \ \mu_{min}} + \frac{\tau_s \gamma_\infty^2 ||\nabla \cdot \mathbb{C} \nabla^s \boldsymbol{u}||_\infty^2}{\gamma_0^2}\right) ||q||^2$$

$$
\begin{aligned}
&- \frac{1}{4}||\boldsymbol{w}||^2 + \frac{\mu_{min}}{4}||\nabla^s \boldsymbol{v}||^2 \\
&+ \left[\frac{3\ \mu_{min}}{4} - \frac{\tau_s c_2^2 \mid \mathbb{C} \mid_s^2}{\gamma_0^2}\left(||\nabla \cdot (\mu \mathbb{C} \nabla^s \boldsymbol{u})||_\infty^2 + \gamma_\infty^2 ||\nabla \mu||_\infty^2\right)\right] ||\nabla^s \boldsymbol{w}||^2 \\
&- \frac{\tau_s \gamma_0^2}{4}||\nabla q||^2 + \tau_s ||\mathbb{C}\nabla^s \boldsymbol{u} \cdot \nabla q||^2 - \frac{\tau_s \gamma_0^2}{2\gamma_\infty^2}||\mathbb{C}\nabla^s \boldsymbol{u} \cdot \nabla q||^2. \qquad (41)
\end{aligned}
$$

Given that τ_s satisfies the inequality (34), we have the coefficient of the term $||\nabla^s \boldsymbol{w}||^2$ in (41) greater than $\frac{\mu_{min}}{4}$. Using this and (33) on the right hand side, we arrive at,

$$
A_s(\{\boldsymbol{v}, q, \boldsymbol{w}\}, \{\boldsymbol{w}, q, \boldsymbol{v}\}) + K_1||\boldsymbol{v}||^2 + K_2||q||^2 + K_3||\boldsymbol{w}||^2 \geq \frac{\mu_{min}}{4}||\nabla^s \boldsymbol{v}||^2 + \frac{\tau_s \gamma_0^2}{4}||\nabla q||^2 + \frac{\mu_{min}}{4}||\nabla^s \boldsymbol{w}||^2 \qquad (42)
$$

where K_1, K_2, and K_3 are given by (36) , (37), and (38), respectively. □

In the following Lemma we prove the continuity of A_s. This result is then used in Corollary 3.4 and Lemma 3.5.

Lemma 3.3. (Continuity of A_s) *For any $\{\boldsymbol{v}_1, q_1, \boldsymbol{w}_1\}$ and $\{\boldsymbol{v}_2, q_2, \boldsymbol{w}_2\} \in \mathcal{W} \times Q \times \mathcal{W}$,*

$$
\begin{aligned}
A_s(\{\boldsymbol{v}_1, q_1, \boldsymbol{w}_1\}, \{\boldsymbol{v}_2, q_2, \boldsymbol{w}_2\}) \leq\ & \epsilon_1||\boldsymbol{v}_2||^2 + (\epsilon_2 + \epsilon_7 + \epsilon_{10})||q_2||^2 + (\epsilon_4 + \epsilon_6 + \epsilon_8 + \epsilon_{11})||\nabla^s \boldsymbol{v}_2||^2 \\
&+ \tau_s\ \epsilon_9||\nabla q_2||^2 + (\epsilon_3 + \epsilon_5)||\nabla^s \boldsymbol{w}_2||^2 + \frac{1}{4\ \epsilon_1}||\boldsymbol{v}_1||^2 \\
&+ \frac{C_{\boldsymbol{u},\mu,\lambda} \mid \mathbb{C} \mid_s^2}{4}\Bigg[\left(\frac{1}{\epsilon_4} + \frac{1}{\epsilon_5}\right)||q_1||^2 + \left(\frac{1}{\epsilon_6} + \frac{1}{\epsilon_7}\right)||\nabla^s \boldsymbol{w}_1||^2 \\
&+ \left(\frac{\tau_s^2}{\epsilon_8} + \frac{\tau_s}{\epsilon_9} + \frac{\tau_s^2}{\epsilon_{10}} + \frac{\tau_s^2}{\epsilon_{11}}\right)||\nabla q_1||^2 + \left(\frac{1}{\epsilon_2} + \frac{1}{\epsilon_3}\right)||\nabla^s \boldsymbol{v}_1||^2\Bigg], \qquad (43)
\end{aligned}
$$

where, $\epsilon_i \in \mathbb{R}^+$, and $C_{\boldsymbol{u},\mu,\lambda}$ is given by,

$$
\begin{aligned}
C_{\boldsymbol{u},\mu,\lambda} =\ & \max\{||\nabla^s \boldsymbol{\lambda}||_\infty^2,\ ||\nabla^s \boldsymbol{u}||_\infty^2,\ \mu_\infty^2,\ c_2^2\ (||\nabla \cdot (\mu \mathbb{C} \nabla^s \boldsymbol{u})||_\infty)^2, \\
& c_2^4\ \mid \mathbb{C} \mid_s^2\ ||\nabla^s \boldsymbol{u}||_\infty^4,\ c_2^2\ (||\nabla \cdot \mathbb{C}\nabla^s \boldsymbol{u}||_\infty)^2,\ c_2^4\ \mid \mathbb{C} \mid_s^2 ||\nabla^s \boldsymbol{u}||_\infty^2 ||\nabla \mu||_\infty^2\}, \qquad (44)
\end{aligned}
$$

Proof. Using definition of A_s (23), we have,

$$
\begin{aligned}
A_s(\{\boldsymbol{v}_1, q_1, \boldsymbol{w}_1\}, \{\boldsymbol{v}_2, q_2, \boldsymbol{w}_2\}) \leq\ & \mid (\boldsymbol{v}_2, \boldsymbol{v}_1) \mid + \mid a(\boldsymbol{\lambda}, \boldsymbol{v}_1; q_2) \mid + \mid a(\boldsymbol{w}_2, \boldsymbol{v}_1; \mu) \mid + \mid a(\boldsymbol{\lambda}, \boldsymbol{v}_2; q_1) \mid \\
&+ \mid a(\boldsymbol{w}_2, \boldsymbol{u}; q_1) \mid + \mid a(\boldsymbol{w}_1, \boldsymbol{v}_2; \mu) \mid + \mid a(\boldsymbol{w}_1, \boldsymbol{u}; q_2) \mid \\
&+ \tau_s \mid (\mathbb{C}\nabla \boldsymbol{v}_2 \cdot \nabla q_1, \nabla \cdot (\mu \mathbb{C} \nabla \boldsymbol{u})) \mid + \tau_s \mid (\mathbb{C}\nabla \boldsymbol{u} \cdot \nabla q_1, \nabla q_2 \cdot \mathbb{C}\nabla \boldsymbol{u}) \mid \\
&+ \tau_s \mid (\mathbb{C}\nabla \boldsymbol{u} \cdot \nabla q_1 \cdot q_2(\nabla \cdot \mathbb{C}\nabla \boldsymbol{u})) \mid + \tau_s \mid (\mathbb{C}\nabla \boldsymbol{u} \cdot \nabla q_1, \nabla \mu \cdot \mathbb{C}\nabla \boldsymbol{v}_2) \mid . \qquad (45)
\end{aligned}
$$

We use Young's inequality for each of the terms on the right hand side, the definition of norm induced by 4^{th} order tensor $\mathbb{C}$ (30) in all the terms except term 1, and the equivalence of norms (29) in terms 8, 9, 10, and 11, to get,

$$
A_s(\{\boldsymbol{v}_1, q_1, \boldsymbol{w}_1\}, \{\boldsymbol{v}_2, q_2, \boldsymbol{w}_2\}) \leq \epsilon_1||\boldsymbol{v}_2||^2 + \frac{1}{4\ \epsilon_1}||\boldsymbol{v}_1||^2 + \epsilon_2||q_2||^2 + \frac{\mid \mathbb{C} \mid_s^2\ ||\nabla^s \boldsymbol{\lambda}||_\infty^2}{4\ \epsilon_2}||\nabla^s \boldsymbol{v}_1||^2 + \epsilon_3||\nabla^s \boldsymbol{w}_2||^2
$$

$$
\begin{aligned}
&+ \frac{\mid \mathbb{C} \mid_s^2 \, \mu_\infty^2}{4 \, \epsilon_3} \|\nabla^s \boldsymbol{v}_1\|^2 + \epsilon_4 \|\nabla^s \boldsymbol{v}_2\|^2 \; + \frac{\mid \mathbb{C} \mid_s^2 \, \|\nabla^s \boldsymbol{\lambda}\|_\infty^2}{4 \, \epsilon_4} \|q_1\|^2 + \epsilon_5 \|\nabla^s \boldsymbol{w}_2\|^2 \\
&+ \frac{\mid \mathbb{C} \mid_s^2 \, \|\nabla^s \boldsymbol{u}\|_\infty^2}{4 \, \epsilon_5} \|q_1\|^2 + \epsilon_6 \|\nabla^s \boldsymbol{v}_2\|^2 + \frac{\mid \mathbb{C} \mid_s^2 \, \mu_\infty^2}{4 \, \epsilon_6} \|\nabla^s \boldsymbol{w}_1\|^2 + \epsilon_7 \|q_2\|^2 \\
&+ \frac{\mid \mathbb{C} \mid_s^2 \, \|\nabla^s \boldsymbol{u}\|_\infty^2}{4 \, \epsilon_7} \|\nabla^s \boldsymbol{w}_1\|^2 \\
&+ \epsilon_8 \|\nabla^s \boldsymbol{v}_2\|^2 + \frac{\tau_s^2 \|\nabla \cdot (\mu \mathbb{C} \nabla^s \boldsymbol{u})\|_\infty^2 c_2^2 \mid \mathbb{C} \mid_s^2}{4 \, \epsilon_8} \|\nabla q_1\|^2 \\
&+ \tau_s \, \epsilon_9 \, \|\nabla q_2\|^2 + \frac{\tau_s c_2^4 \mid \mathbb{C} \mid_s^4 \, \|\nabla^s \boldsymbol{u}\|_\infty^4}{4 \, \epsilon_9} \|\nabla q_1\|^2 \\
&+ \epsilon_{10} \, \|q_2\|^2 + \frac{\tau_s^2 \|\nabla \cdot \mathbb{C} \nabla^s \boldsymbol{u}\|_\infty^2 c_2^2 \mid \mathbb{C} \mid_s^2}{4 \, \epsilon_{10}} \|\nabla q_1\|^2 \\
&+ \epsilon_{11} \, \|\nabla^s \boldsymbol{v}_2\|^2 + \frac{\tau_s^2 c_2^4 \mid \mathbb{C} \mid_s^4 \, \|\nabla^s \boldsymbol{u}\|_\infty^2 \|\nabla \mu\|_\infty^2}{4 \, \epsilon_{11}} \|\nabla q_1\|^2,
\end{aligned} \tag{46}
$$

where ϵ_i are positive constants. By collecting the coefficients of similar terms and introducing the constant $C_{\boldsymbol{u},\mu,\lambda}$ as defined in (44), we prove the desired result. □

Corollary 3.4. *For any $\{\boldsymbol{v}_1, q_1, \boldsymbol{w}_1\}$, and $\{\boldsymbol{v}_2, q_2, \boldsymbol{w}_2\}$, we have,*

$$
A_s(\{\boldsymbol{v}_1, q_1, \boldsymbol{w}_1\}, \{\boldsymbol{v}_2, q_2, \boldsymbol{w}_2\}) \leq \frac{1}{8} \, |\!|\!| \{\boldsymbol{v}_2, q_2, \boldsymbol{w}_2\} |\!|\!|^2 + \|\boldsymbol{v}_2\|^2 + \mu_{min} \, \|q_2\|^2 + f(\{\boldsymbol{v}_1, q_1, \boldsymbol{w}_1\}), \tag{47}
$$

where,

$$
\begin{aligned}
f(\{\boldsymbol{v}_1, q_1, \boldsymbol{w}_1\}) \equiv & \frac{1}{4} \|\boldsymbol{v}_1\|^2 + \frac{C_{\boldsymbol{u},\mu,\lambda} \mid \mathbb{C} \mid_s^2}{4} \Big[\frac{48}{\mu_{min}} \|q_1\|^2 + \frac{35}{\mu_{min}} \|\nabla^s \boldsymbol{w}_1\|^2 \\
& + \left(\frac{67 \, \tau_s^2}{\mu_{min}} + \frac{8 \, \tau_s}{\gamma_0^2} \right) \|\nabla q_1\|^2 + \frac{19}{\mu_{min}} \|\nabla^s \boldsymbol{v}_1\|^2 \Big].
\end{aligned} \tag{48}
$$

Proof. We make the following choices for ϵ in (43),

$$
\epsilon_1 = 1, \epsilon_2 = \epsilon_7 = \epsilon_{10} = \frac{\mu_{min}}{3}, \epsilon_3 = \epsilon_5 = \frac{\mu_{min}}{16}, \epsilon_4 = \epsilon_6 = \epsilon_8 = \epsilon_{11} = \frac{\mu_{min}}{32}, \epsilon_9 = \frac{\gamma_0^2}{8}, \tag{49}
$$

and use the definition of $|\!|\!| \cdot |\!|\!|$ (36), to arrive at the desired result. □

Next we apply Aubin-Nitsche duality argument [23, 24, 25, 22] to this problem. This argument is a result that bounds the L_2 error by H^1 like error terms that are multiplied by the mesh size. It is useful in proving convergence for variational forms where coercivity can only be obtained by adding additional L_2 terms (Garding's Inequality). We prepare for it by considering the following adjoint problem, which is driven by total error in the field $\{\delta \boldsymbol{u}, \delta\mu, \delta\boldsymbol{\lambda}\}$.
Find $\{\boldsymbol{u}_a, \mu_a, \boldsymbol{\lambda}_a\} \in \mathcal{W} \times \mathcal{Q} \times \mathcal{W}$, such that, $\forall \{\boldsymbol{v}, q, \boldsymbol{w}\} \in \mathcal{W} \times \mathcal{Q} \times \mathcal{W}$

$$
A_s(\{\boldsymbol{\lambda}_a, \mu_a, \boldsymbol{u}_a\}, \{\boldsymbol{v}, q, \boldsymbol{w}\}) = (\boldsymbol{e}_{\boldsymbol{u}}, \boldsymbol{v}) + (e_\mu, q) + (\boldsymbol{e}_\lambda, \boldsymbol{w}). \tag{50}
$$

The Euler-Lagrange form (strong form) for the above adjoint equation is obtained by performing integration-by-parts on the terms on the left hand side so that all the derivatives from the weighting functions are transferred to the adjoint field. By doing so, we obtain the following strong form of the adjoint problem: Find $\boldsymbol{u}_a$, μ_a, and $\boldsymbol{\lambda_a}$, such that,

$$\boldsymbol{\lambda}_a - \nabla \cdot (\mu_a \mathbb{C} \nabla \boldsymbol{\lambda}) - \nabla \cdot (\mu \mathbb{C} \nabla \boldsymbol{u}_a) - \tau_s (\nabla \cdot (\mathbb{C}(\nabla \cdot (\mu \mathbb{C} \nabla \boldsymbol{u}) \otimes \nabla \mu_a))) = \boldsymbol{e_u}, \tag{51}$$

$$\nabla \boldsymbol{\lambda} \cdot \mathbb{C} \nabla \boldsymbol{\lambda}_a + \nabla \boldsymbol{u}_a : \mathbb{C} \nabla \boldsymbol{u} - \tau_s \nabla \cdot (\mathbb{C} \nabla \boldsymbol{u} \cdot (\mathbb{C} \boldsymbol{u} \cdot \nabla \mu_a)) + \tau_s (\mathbb{C} \nabla \boldsymbol{u} \cdot \nabla \mu_a) \cdot (\nabla \cdot \mathbb{C} \nabla \boldsymbol{u}) = e_\mu, \tag{52}$$

$$\nabla \cdot (\mu \mathbb{C} \boldsymbol{\lambda}_a) + \nabla \cdot (\mu_a \mathbb{C} \boldsymbol{u}) = -\boldsymbol{e}_\lambda, \quad \text{in } \Omega, \tag{53}$$

$$(\mathbb{C} \nabla \boldsymbol{u} \cdot \nabla \mu_a) \cdot (\mathbb{C} \nabla \boldsymbol{u} \cdot \hat{\boldsymbol{n}}) = 0, \quad \text{on} \quad \partial\Omega. \tag{54}$$

In (54), $\hat{\boldsymbol{n}}$ represents outward unit normal on the boundary ($\partial\Omega$) of Ω. For these adjoint fields, we assume that the following regularity estimate hold.

$$\|\nabla^2 \boldsymbol{u}_a\| \leq C_R \left(\|\boldsymbol{e_u}\| + \|\boldsymbol{e}_\mu\| + \|\boldsymbol{e}_\lambda\| \right), \tag{55}$$

$$\|\nabla^2 \mu_a\| \leq C_R \left(\|\boldsymbol{e_u}\| + \frac{\|\boldsymbol{e}_\mu\|}{\tau_s} + \|\boldsymbol{e}_\lambda\| \right), \tag{56}$$

$$\|\nabla^2 \boldsymbol{\lambda}_a\| \leq C_R \left(\|\boldsymbol{e_u}\| + \|\boldsymbol{e}_\mu\| + \|\boldsymbol{e}_\lambda\| \right) \tag{57}$$

where, C_R is the regularity constant. We note that proving these estimates in general is beyond the scope of this analysis. However in Appendix A, for a simple, representative case in one dimension, we have demonstrated that these estimates are reasonable. Along with these regularity estimates, we will also use the following interpolation estimates [26, 27] in Lemma 3.5:

$$\|\boldsymbol{\eta}_{\boldsymbol{u}_a}\| \leq C_I h^2 \|\nabla^2 \boldsymbol{u}_a\|, \qquad \|\nabla \boldsymbol{\eta}_{\boldsymbol{u}_a}\| \leq C_I h \|\nabla^2 \boldsymbol{u}_a\|, \tag{58}$$

$$\|\boldsymbol{\eta}_{\mu_a}\| \leq C_I h^2 \|\nabla^2 \mu\|, \qquad \|\nabla \boldsymbol{\eta}_{\mu_a}\| \leq C_I h \|\nabla^2 \mu_a\|, \tag{59}$$

$$\|\boldsymbol{\eta}_{\lambda_a}\| \leq C_I h^2 \|\nabla^2 \boldsymbol{\lambda}_a\|, \qquad \|\nabla \boldsymbol{\eta}_{\lambda_a}\| \leq C_I h \|\nabla^2 \boldsymbol{\lambda}_a\|, \tag{60}$$

where $\{\boldsymbol{\eta}_{\lambda_a}, \eta_{\mu_a}, \boldsymbol{\eta}_{\boldsymbol{u}_a}\}$ represents the error in the interpolant for the adjoint filed, C_I is the interpolation constant, and h is the finite element size. We are now ready to apply the so-called Aubin-Nitsche duality argument to our problem.

Lemma 3.5. (Aubin-Nitsche duality argument) *There exists a constant C_n such that for the finite element approximation errors $\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\} \in \mathcal{W} \times Q \times \mathcal{W}$ to the solution of the problem (22),*

$$\|\boldsymbol{e_u}\|^2 + \|e_\mu\|^2 + \|\boldsymbol{e}_\lambda\|^2 \leq C_n \; h \; \left\vert\kern-0.25ex\left\vert\kern-0.25ex\left\vert \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\} \right\vert\kern-0.25ex\right\vert\kern-0.25ex\right\vert^2. \tag{61}$$

Proof. We choose $\{\boldsymbol{v}, q, \boldsymbol{w}\} \leftarrow \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}$ in (50), to arrive at,

$$\begin{aligned}
\|\boldsymbol{e_u}\|^2 + \|e_\mu\|^2 + \|\boldsymbol{e}_\lambda\|^2 =& A_s(\{\boldsymbol{\lambda}_a, \mu_a, \boldsymbol{u}_a\}, \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}), \\
=& A_s(\{\boldsymbol{\lambda}_a - \boldsymbol{\lambda}_a^h, \mu_a - \mu_a^h, \boldsymbol{u}_a - \boldsymbol{u}_a^h\}, \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}), && \text{(Using (31))} \\
=& A_s(\{\boldsymbol{e}_{\lambda_a}, e_{\mu_a}, \boldsymbol{e}_{\boldsymbol{u}_a}\}, \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}), \\
=& A_s(\{\boldsymbol{e}^h_{\lambda_a} + \boldsymbol{\eta}_{\lambda_a}, e^h_{\mu_a} + \eta_{\mu_a}, \boldsymbol{e}^h_{\boldsymbol{u}_a} + \boldsymbol{\eta}_{\boldsymbol{u}_a}\}, \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}), && \text{(Using (31))} \\
=& A_s(\{\boldsymbol{\eta}_{\lambda_a}, \eta_{\mu_a}, \boldsymbol{\eta}_{\boldsymbol{u}_a}\}, \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}).
\end{aligned}$$

Where we have decomposed the error in the adjoint field as $\{\boldsymbol{e}_{\lambda_a}, e_{\mu_a}, \boldsymbol{e}_{\boldsymbol{u}_a}\} = \{\boldsymbol{e}^h_{\lambda_a} + \boldsymbol{\eta}_{\lambda_a}, e^h_{\mu_a} + \eta_{\mu_a}, \boldsymbol{e}^h_{\boldsymbol{u}_a} + \boldsymbol{\eta}_{\boldsymbol{u}_a}\}$. In this decomposition $\{\boldsymbol{e}^h_{\lambda_a}, e^h_{\mu_a}, \boldsymbol{e}^h_{\boldsymbol{u}_a}\} \in \mathcal{W} \times \mathcal{Q} \times \mathcal{W}$ is the error associated with the method, and $\{\boldsymbol{\eta}_{\lambda_a}, \eta_{\mu_a}, \boldsymbol{\eta}_{\boldsymbol{u}_a}\}$ is the interpolation error. Further, using Lemma 3.3, in the equation above we arrive at,

$$
\begin{aligned}
||\boldsymbol{e_u}||^2 + ||e_\mu||^2 + ||\boldsymbol{e}_\lambda||^2 \leq & \epsilon_1 ||\boldsymbol{e_u}||^2 + (\epsilon_2 + \epsilon_7 + \epsilon_{10})||e_\mu||^2 + (\epsilon_4 + \epsilon_6 + \epsilon_8 + \epsilon_{11})||\nabla^s \boldsymbol{e_u}||^2 \\
& + \tau_s\ \epsilon_9 ||\nabla e_\mu||^2 + (\epsilon_3 + \epsilon_5)||\nabla^s \boldsymbol{e}_\lambda||^2 + \frac{1}{4\ \epsilon_1}||\boldsymbol{\eta}_{\lambda_a}||^2 \\
& + \frac{C_{\boldsymbol{u},\mu,\lambda}\mathbb{C}_m^2}{4}\Bigg[\left(\frac{1}{\epsilon_4} + \frac{1}{\epsilon_5}\right)||\eta_{\mu_a}||^2 + \left(\frac{1}{\epsilon_6} + \frac{1}{\epsilon_7}\right)||\nabla^s \boldsymbol{\eta}_{\boldsymbol{u}_a}||^2 \\
& + \left(\frac{\tau_s^2}{\epsilon_8} + \frac{\tau_s}{\epsilon_9} + \frac{\tau_s^2}{\epsilon_{10}} + \frac{\tau_s^2}{\epsilon_{11}}\right)||\nabla \eta_{\mu_a}||^2 + \left(\frac{1}{\epsilon_2} + \frac{1}{\epsilon_3}\right)||\nabla^s \boldsymbol{\eta}_{\lambda_a}||^2\Bigg], \qquad (62) \\
\leq & \epsilon_1 ||\boldsymbol{e_u}||^2 + (\epsilon_2 + \epsilon_7 + \epsilon_{10})||e_\mu||^2 + (\epsilon_4 + \epsilon_6 + \epsilon_8 + \epsilon_{11})||\nabla^s \boldsymbol{e_u}||^2 \qquad (63) \\
& + \tau_s\ \epsilon_9 ||\nabla e_\mu||^2 + (\epsilon_3 + \epsilon_5)||\nabla^s \boldsymbol{e}_\lambda||^2 + \frac{h^4\ C_I^2}{4\ \epsilon_1}||\nabla^2 \boldsymbol{\lambda}_a||^2 \\
& + \frac{C_{\boldsymbol{u},\mu,\lambda} \mid \mathbb{C} \mid_s^2 C_I^2}{4}\Bigg[\left(\frac{1}{\epsilon_4} + \frac{1}{\epsilon_5}\right)\ h^4\ ||\nabla^2 \mu_a||^2 + \left(\frac{1}{\epsilon_6} + \frac{1}{\epsilon_7}\right)\ h^2\ ||\nabla^2 \boldsymbol{u}_a||^2 \\
& + \left(\frac{\tau_s^2}{\epsilon_8} + \frac{\tau_s}{\epsilon_9} + \frac{\tau_s^2}{\epsilon_{10}} + \frac{\tau_s^2}{\epsilon_{11}}\right)\ h^2\ ||\nabla^2 \mu_a||^2 + \left(\frac{1}{\epsilon_2} + \frac{1}{\epsilon_3}\right)\ h^2\ ||\nabla^2 \boldsymbol{\lambda}_a||^2\Bigg], \\
& \text{(Interpolation estimates (58) to (60))} \\
\leq & \epsilon_1 ||\boldsymbol{e_u}||^2 + (\epsilon_2 + \epsilon_7 + \epsilon_{10})\ ||e_\mu||^2 + (\epsilon_4 + \epsilon_6 + \epsilon_8 + \epsilon_{11})||\nabla^s \boldsymbol{e_u}||^2 \\
& + \tau_s\ \epsilon_9 ||\nabla e_\mu||^2 + (\epsilon_3 + \epsilon_5)||\nabla^s \boldsymbol{e}_\lambda||^2 + \frac{3h^4\ C_I^2\ C_R^2}{4\ \epsilon_1}\left(||\boldsymbol{e_u}||^2 + ||\boldsymbol{e}_\mu||^2 + ||\boldsymbol{e}_\lambda||^2\right) \\
& + \frac{3\ C_{\boldsymbol{u},\mu,\lambda} \mid \mathbb{C} \mid_s^2 C_I^2 C_R^2}{4}\Bigg[\left(\frac{h^4}{\epsilon_4} + \frac{h^4}{\epsilon_5}\right)\left(||\boldsymbol{e_u}||^2 + \frac{||\boldsymbol{e}_\mu||^2}{\tau_s^2} + ||\boldsymbol{e}_\lambda||^2\right) \\
& + \left(\frac{h^2}{\epsilon_6} + \frac{h^2}{\epsilon_7}\right)\left(||\boldsymbol{e_u}||^2 + ||\boldsymbol{e}_\mu||^2 + ||\boldsymbol{e}_\lambda||^2\right) \\
& + \left(\frac{\tau_s^2 h^2}{\epsilon_8} + \frac{\tau_s h^2}{\epsilon_9} + \frac{\tau_s^2\ h^2}{\epsilon_{10}} + \frac{\tau_s^2\ h^2}{\epsilon_{11}}\right)\left(||\boldsymbol{e_u}||^2 + \frac{||\boldsymbol{e}_\mu||^2}{\tau_s^2} + ||\boldsymbol{e}_\lambda||^2\right) \\
& + \left(\frac{h^2}{\epsilon_2} + \frac{h^2}{\epsilon_3}\right)\left(||\boldsymbol{e_u}||^2 + ||\boldsymbol{e}_\mu||^2 + ||\boldsymbol{e}_\lambda||^2\right)\Bigg]. \qquad \text{(Regularity estimates (55))}
\end{aligned}
$$

We make the following choices for ϵ_i in the above expression,

$$
\epsilon_1 = \epsilon_2 = \epsilon_7 = \epsilon_{10} = h, \epsilon_3 = \epsilon_5 = \frac{\mu_{min}\ h}{2}, \epsilon_4 = \epsilon_6 = \epsilon_8 = \epsilon_{11} = \frac{\mu_{min}\ h}{4}, \epsilon_9 = \gamma_0^2\ h, \qquad (64)
$$

to arrive at,

$$
\begin{aligned}
||\boldsymbol{e_u}||^2 + ||e_\mu||^2 + ||\boldsymbol{e}_\lambda||^2 \leq & D_1 ||\boldsymbol{e_u}||^2 + D_2 ||e_\mu||^2 + D_3 ||\boldsymbol{e}_\lambda||^2 \\
& + h\left[\mu_{min}(\ ||\nabla^s \boldsymbol{e}_\lambda||^2 + ||\nabla^s \boldsymbol{e_u}||^2) + \tau_s \gamma_0^2\ ||\nabla e_\mu||^2\right], \qquad (65)
\end{aligned}
$$

where,

$$D_1 = h\left[1 + \frac{3h^2C_I^2C_R^2}{4} + \frac{3C_{\boldsymbol{u},\mu,\lambda} \mid \mathbb{C} \mid_s^2 C_I^2C_R^2}{4}\left(\frac{6h^2}{\mu_{min}} + \frac{6}{\mu_{min}} + \frac{8\tau_s^2}{\mu_{min}} + \frac{\tau_s}{\gamma_0^2} + \tau_s^2 + 2\right)\right], \tag{66}$$

$$D_2 = h\left[3 + \frac{3h^2C_I^2C_R^2}{4} + \frac{3C_{\boldsymbol{u},\mu,\lambda} \mid \mathbb{C} \mid_s^2 C_I^2C_R^2}{4}\left(\frac{6h^2}{\tau_s^2\mu_{min}} + \frac{14}{\mu_{min}} + \frac{1}{\gamma_0^2\tau_s} + 3\right)\right], \tag{67}$$

$$D_3 = D_1 - h. \tag{68}$$

The coefficients D_1, D_2, and D_3, that multiply the $||\boldsymbol{e_u}||$, $||\boldsymbol{e}_\mu||$ and $||\boldsymbol{e}_\lambda||$ terms respectively, on right hand side of (65), are $O(h)$. Therefore for sufficiently small h, we can hide these terms on the left hand side. That is we select h in (65), such that, $D_1 \le \frac{1}{2}$, $D_2 \le \frac{1}{2}$, $D_3 \le \frac{1}{2}$, and we have,

$$||\boldsymbol{e_u}||^2 + ||e_\mu||^2 + ||\boldsymbol{e}_\lambda||^2 \le C_n\ h\ \left|\!\left|\!\left|\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}\right|\!\right|\!\right|^2,\ \text{with } C_n = 2. \tag{69}$$

□

We are now ready to prove the main result of this analysis.

Theorem 3.6. *There exists a positive constant C, independent of h, such that,*

$$\left|\!\left|\!\left|\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}\right|\!\right|\!\right|^2 \le C(||\boldsymbol{\eta_u}||^2 + ||\eta_\mu||^2 + ||\nabla^s\boldsymbol{\eta_u}||^2 + \left(\tau_s^2 + \tau_s\right)||\nabla\eta_\mu||^2 + ||\nabla^s\boldsymbol{\eta}_\lambda||^2),$$

where $\{\boldsymbol{\eta_u}, \eta_\mu, \boldsymbol{\eta}_\lambda\}$ *are errors in the interpolant for* $\{\boldsymbol{u}, \mu, \boldsymbol{\lambda}\}$.

Proof. We choose $\{\boldsymbol{v}, q, \boldsymbol{w}\} \leftarrow \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}$ in Lemma (3.2), to arrive at,

$$\begin{aligned}
\frac{1}{4}\left|\!\left|\!\left|\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}\right|\!\right|\!\right|^2 \le& A_s(\{\boldsymbol{e}_\lambda, e_\mu, \boldsymbol{e_u}\}, \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}) + K_1||\boldsymbol{e_u}||^2 + K_2||e_\mu||^2 + K_3||\boldsymbol{e}_\lambda||^2,\\
\le& A_s(\{\boldsymbol{e}_\lambda^h + \boldsymbol{\eta}_\lambda, e_\mu^h + \eta_\mu, \boldsymbol{e}_{\boldsymbol{u}}^h + \boldsymbol{\eta_u}\}, \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}) + K_1||\boldsymbol{e_u}||^2 + K_2||e_\mu||^2 + K_3||\boldsymbol{e}_\lambda||^2,\\
&\text{(Using defintion of total error)}\\
\le& A_s(\{\boldsymbol{\eta}_\lambda, \eta_\mu, \boldsymbol{\eta_u}\}, \{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}) + K_1||\boldsymbol{e_u}||^2 + K_2||e_\mu||^2 + K_3||\boldsymbol{e}_\lambda||^2,\\
&\text{(Orthogonality of } A_s\text{, (31))}\\
\le& \frac{1}{8}\left|\!\left|\!\left|\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}\right|\!\right|\!\right|^2 + f(\{\boldsymbol{\eta_u}, \eta_\mu, \eta_\lambda\}) + (K_1+1)||\boldsymbol{e_u}||^2 + K_3\ ||e_\lambda||^2\\
&+ (K_2 + \mu_{min})\ ||e_\mu||^2, \qquad \text{(Corollary (3.4))}\\
\frac{1}{8}\left|\!\left|\!\left|\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}\right|\!\right|\!\right|^2 \le& f(\{\boldsymbol{\eta_u}, \eta_\mu, \boldsymbol{\eta}_\lambda\}) + K_{max}(||\boldsymbol{e_u}||^2 + ||e_\mu||^2 + ||e_\lambda||^2),
\end{aligned} \tag{70}$$

where, $K_{max} = \max\{K_1 + 1, K_2 + \mu_{min}, K_3\}$. Further, by using Lemma (3.5), we are led to,

$$\frac{1}{8}\left|\!\left|\!\left|\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}\right|\!\right|\!\right|^2 \le f(\{\boldsymbol{\eta_u}, \eta_\mu, \boldsymbol{\eta}_\lambda\}) + K_{max}\ \mathbb{C}_n\ h\ \left|\!\left|\!\left|\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}\right|\!\right|\!\right|^2. \tag{71}$$

So, for sufficiently small h such that, $h \le \frac{1}{16\ K_{max}\ C_n}$, we have,

$$\frac{1}{16}\left|\!\left|\!\left|\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e}_\lambda\}\right|\!\right|\!\right|^2 \le f(\{\boldsymbol{\eta_u}, \eta_\mu, \boldsymbol{\eta}_\lambda\}), \tag{72}$$

where,

$$f(\{\boldsymbol{\eta_u}, \eta_\mu, \boldsymbol{\eta_\lambda}\}) \equiv \frac{1}{4}||\boldsymbol{\eta_u}||^2 + \frac{C_{\boldsymbol{u},\mu,\lambda} \mid \mathbb{C} \mid_s^2}{4}\Big[\frac{48}{\mu_{min}}||\eta_\mu||^2 + \frac{35}{\mu_{min}}||\nabla^s \boldsymbol{\eta_\lambda}||^2 + \left(\frac{67\ \tau_s^2}{\mu_{min}} + \frac{8\ \tau_s}{\gamma_0^2}\right)||\nabla\eta_\mu||^2 + \frac{19}{\mu_{min}}||\nabla^s \boldsymbol{\eta_u}||^2\Big]. \tag{73}$$

Finally we choose a constant C in (73), independent of h, such that,

$$C = \max\left\{\frac{1}{4}, \frac{12C_{\boldsymbol{u},\mu,\lambda} \mid \mathbb{C} \mid_s^2}{\mu_{min}}, \frac{C_{\boldsymbol{u},\mu,\lambda} \mid \mathbb{C} \mid_s^2}{4}\left(\frac{67\ \tau_s^2}{\mu_{min}} + \frac{8\ \tau_s}{\gamma_0^2}\right)\right\}. \tag{74}$$

to obtain the desired result,

$$\left|\left|\left|\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e_\lambda}\}\right|\right|\right|^2 \leq C(||\boldsymbol{\eta_u}||^2 + ||\eta_\mu||^2 + ||\nabla^s \boldsymbol{\eta_u}||^2 + \left(\tau_s^2 + \tau_s\right)||\nabla\eta_\mu||^2 + ||\nabla^s \boldsymbol{\eta_\lambda}||^2),$$

□

Remark1. Assuming τ_s is positive and independent of h, from Theorem 3.6 we conclude that for equal order interpolation of order p,

$$\left|\left|\left|\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e_\lambda}\}\right|\right|\right| \leq C_1 h^p. \tag{75}$$

Recognizing that norm $\left|\left|\left|\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e_\lambda}\}\right|\right|\right|$ is comprised of the gradients of $\{\boldsymbol{e_u}, e_\mu, \boldsymbol{e_\lambda}\}$ (see (35)), we note that this rate of convergence is optimal.

Remark2. In this manuscript we have considered the inverse plane stress elasticity problem and the inverse heat conduction problem. For these we have proposed a finite element method that converges at optimal rates. For its stability, the proposed method relies on the stability of the forward elliptic problem (for the state and the adjoint variables) and on the residual based stabilization term (for the optimization variable). This approach can be applied/extended to other inverse elliptic problems including plane strain and three-dimensional elasticity, inverse Helmholtz equation, and inverse electrostatics and electrodynamics. These extensions will be considered in future work.

4. Numerical Verification

We consider two numerical examples that demonstrate that with mesh refinement the proposed method converges at optimal rates. In the first problem, we consider a two-dimensional plane stress problem on a unit square domain, with dimensions $\Omega \equiv [0, 1] \times [0, 1]$. The equillibrium equation for two-dimensional plane stress is given by,

$$\nabla \cdot (\mu \boldsymbol{E}) = 0, \tag{76}$$

where,

$$\mathbf{E} = \begin{bmatrix} 2\epsilon_{11} + \epsilon_{22} & \epsilon_{12} \\ \epsilon_{12} & \epsilon_{11} + 2\epsilon_{22} \end{bmatrix}.$$

We consider the following strain tensor $\boldsymbol{E}$,

$$\mathbf{E} = \begin{bmatrix} \frac{x-y}{2} + 0.2 & \frac{y-x}{2} - 0.7 \\ \frac{y-x}{2} - 0.7 & \frac{x-y}{2} + 1.2 \end{bmatrix}.$$

The displacement field, $\tilde{\boldsymbol{u}} = [\tilde{u}_1, \tilde{u}_2]^T$, that generates the above strain field is given by,

$$\tilde{u}_1 = \frac{1}{3}(0.25x^2 + 1.225y^2 - 0.5xy - 0.8x - 2.1y + \alpha_x), \qquad \text{where, } \alpha_x \in \mathbb{R} \tag{77}$$

$$\tilde{u}_2 = \frac{1}{3}(-0.25x^2 - 1.25y^2 + 0.5xy + 2.2x - 2.1y + \alpha_y), \qquad \text{where, } \alpha_y \in \mathbb{R}. \tag{78}$$

We select $\alpha_x = 3$, and $\alpha_y = 3$, so that $\tilde{\boldsymbol{u}}(0,0) = [1,1]^T$. The shear modulus distribution that is compatible with this displacement field is $\mu = e^{2(x+y)}$.

To solve the problem, the domain (Ω) is discretized by generating a mesh of uniform, bilinear, square elements (Ω_e). For the inverse problem, the boundary conditions are such that the shear modulus, μ, is prescribed at one point (top-right corner) in Ω, the displacement field is set equal to the measured displacement, and the Lagrange multiplier(λ) is prescribed to be zero everywhere on the boundary of domain Ω. The non-linear saddle point problem is solved using a Newton-like method described in (22). The initial guess for μ is a constant, for $\tilde{\boldsymbol{u}}$ it is the displacement field consistent with a constant μ and imposed BC's, and for $\boldsymbol{\lambda}$ it is $\mathbf{0}$.

The L_2, and H^1 semi-norm of error in the shear modulus field (μ) is evaluated by solving the inverse problem on successively refined meshes. We consider the total number of elements $N_e = 8^2, 16^2, 32^2, 64^2, 128^2, 256^2$ in Ω. The results are plotted in Figure 1 as a function of mesh size. From the figure we observe that the error measured in H^1-seminorm and the L_2 norm converges as h^2. For $p = 1$, this represents the optimal rate for the L_2 norm, and better than optimal rate (super-convergence) for the H^1 semi-norm. We note that the Newton iterations for this problem without any stabilization diverged and did not lead to a meaningful result.

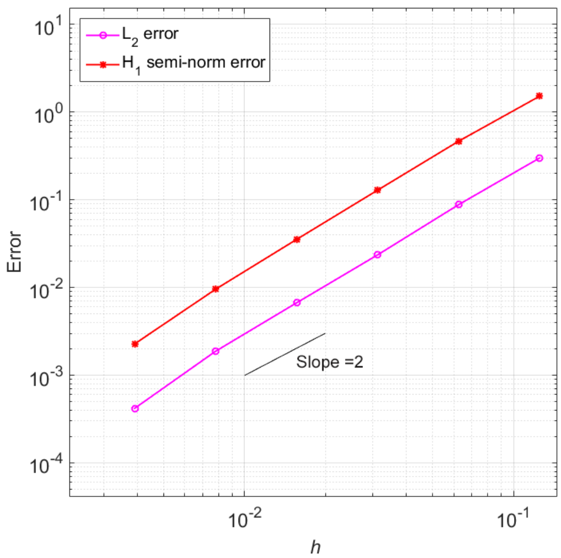
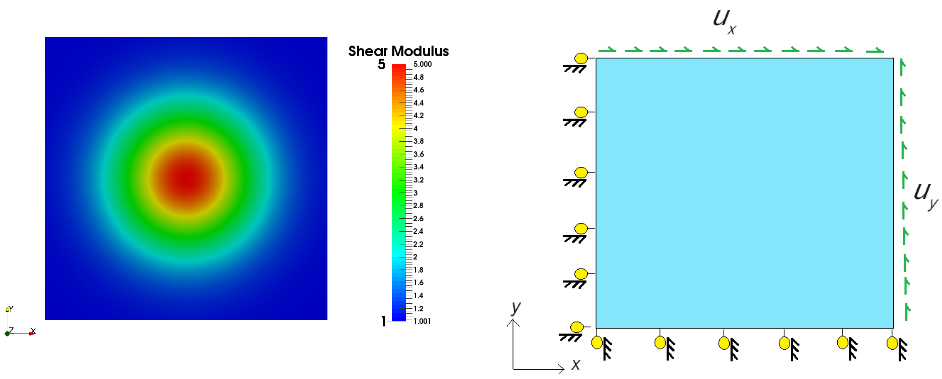


Fig. 1: Error in shear modulus versus mesh size for $\tau_s = 1e-04$

For the second problem we consider a Gaussian distribution for the shear modulus, $\mu = 1 + 4e^{-16((x-0.5)^2+(y-0.5)^2)}$, in $\Omega \equiv [0,1] \times [0,1]$ (see Figure (2a)). The boundary conditions are schematically shown in Figure (2b). The value of $\tilde{u}_y$ on the right edge is given by $\tilde{u}_y = 0.01$, and of the $\tilde{u}_x$ on the right edge is given by $\tilde{u}_x = 0.01$.

The forward problem is solved on a uniform mesh of square finite bilinear finite elements. This solution is used as the reference measured displacement field. This field is downsampled onto coarse mesh and used as measured data for the inverse problem. The inverse problem is solved on meshes varying from 8×8 elements to 128×128 elements. In each case the predicted displacement field is set equal to measured displacement field at the boundaries, while the Lagrange multiplier is set to zero. The non-linear saddle point problem is solved using the iterations described in (22).

The error in the shear modulus, measured in the L_2 norm and H^1 semi-norm, is shown in Figure 3 as a function of mesh size. Noting that the inverse problem is solved using bilinear finite elements, we recognize that both errors converge at optimal rates. In contrast to this the unstabilized problem once again fails to converge at a stable solution.



(a) Shear modulus distribution

(b) Boundary conditions

Fig. 2: Exact shear modulus distribution (left), and boundary conditions for the forward problem (right).

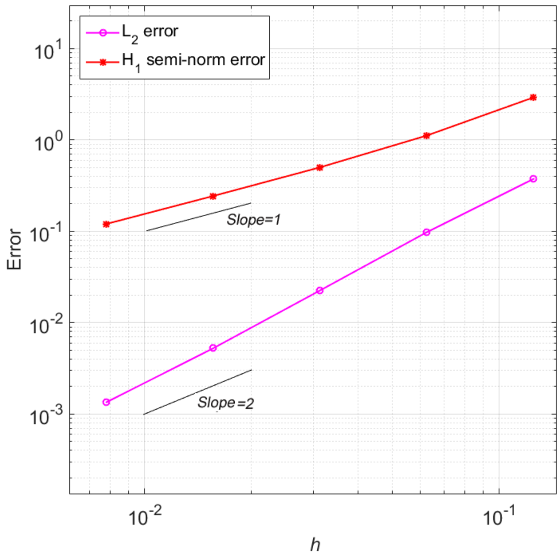


Fig. 3: Error in the shear modulus versus mesh size for $\tau_s = 1e - 04$

5. Conclusions

We consider the solution of the inverse plane stress linear elasticity (or alternately the inverse heat conduction/hydraulic conductivity) problem. we choose a minimization approach to solve this problem by looking for the saddle point of a Lagrangian comprised of an L_2 data mismatch term, and the constraint of the equations of equilibrium. This leads us to a non-linear variational problems. The Galerkin finite element discretization and subsequent linearization (to apply Newton's method) of this equation leads to a saddle point system that lacks stability. We stabilize the system by adding a residual based stabilization term to the variational equations, and the prove stability of the numerical method. We also prove that the erros in the solution field converge with mesh refinement, and demonstrate this convergence using two test problems.

Our approach of constructing a stable system that converges at optimal rates relied on demonstrating stability for the state variables (displacements), the Lagrange multipliers, and the material parameters (shear modulus). For the state variable and the Lagrange multipliers we relied o the stability inherited from the forward elliptic PDE. For the material parameters, we relied on the additional residual-based stabilization term. When viewed from this perspective, this approach can be extended to other elliptic inverse problems such as the inverse plane stress and three-dimensional elasticity, the inverse Helmholtz, and the inverse electrodynamics problems.

Appendix A. Regularity Estimates

We consider the strong form of the adjoint equation (51) to (53) in one spatial dimension,

$$\lambda_a - c\frac{\partial \lambda}{\partial x}\frac{\partial \mu_a}{\partial x} - c\mu\frac{\partial^2 u_a}{\partial^2 x} = e_u, \tag{79}$$

$$c\frac{\partial\lambda}{\partial x}\frac{\partial\lambda_a}{\partial x}+c\frac{\partial u}{\partial x}\frac{\partial u_a}{\partial x}-\tau_s c^2\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2\mu_a}{\partial^2 x}=e_\mu, \tag{80}$$

$$\mu c\frac{\partial^2\lambda_a}{\partial^2 x}+c\frac{\partial u}{\partial x}\frac{\partial\mu_a}{\partial x}=e_\lambda. \tag{81}$$

Further, for simplicity we assume periodic boundary condition. We substitute $\epsilon_u=\frac{\partial u}{\partial x}$, $\epsilon_\mu=\frac{\partial\mu}{\partial x}$, and $\epsilon_\lambda=\frac{\partial\lambda}{\partial x}$, and assume that they are constant. We may then transform (79) to (81) into the Fourier domain, and write,

$$\begin{bmatrix}\mu ck^2 & -ick\epsilon_\lambda & 1\\ ick\epsilon_u & \tau_s c^2k^2\epsilon_u^2 & ick\epsilon_\lambda\\ 0 & -ick\epsilon_u & \mu ck^2\end{bmatrix}\begin{bmatrix}\hat{u}_k\\ \hat{\mu}_k\\ \hat{\lambda}_k\end{bmatrix}=\begin{bmatrix}\hat{e}_u\\ \hat{e}_\mu\\ \hat{e}_\lambda\end{bmatrix}. \tag{82}$$

Where k is the wavenumber, and $\hat{(\cdot)}$ denotes the Fourier coefficient of a variable. We solve (82) to arrive at,

$$\hat{u}_k\propto\left(\frac{\tau_s k^2-1}{1-2k^2+\tau_s k^4}\right)\hat{e}_u+\left(i\frac{1}{1-2k^2+\tau_s k^4}\frac{k^2-1}{k}\right)\hat{e}_\mu+\left(\frac{1-\tau_s}{1-2k^2+\tau_s k^4}\right)\hat{e}_\lambda, \tag{83}$$

$$\hat{\mu}_k\propto\left(\frac{-ik}{1-2k^2+\tau_s k^4}\right)\hat{e}_u+\left(\frac{k^2}{1-2k^2+\tau_s k^4}\right)\hat{e}_\mu+\left(\frac{ik(1-\tau_s k^2)}{1-2k^2+\tau_s k^4}+\frac{i}{k}\right)\hat{e}_\lambda, \tag{84}$$

$$\hat{\lambda}_k\propto\left(\frac{1}{1-2k^2+\tau_s k^4}\right)\hat{e}_u+\left(\frac{ik}{1-2k^2+\tau_s k^4}\right)\hat{e}_\mu+\left(\frac{-1+\tau_s k^2}{1-2k^2+\tau_s k^4}\right)\hat{e}_\lambda. \tag{85}$$

For large values of k, we can deduce,

$$\hat{u}_k\propto\frac{1}{k^2}\hat{e}_u+\frac{i}{\tau_s k^3}\hat{e}_\mu+\frac{1-\tau_s}{\tau_s k^4}\hat{e}_\lambda \tag{86}$$

$$\hat{u}_\mu\propto-\frac{i}{\tau_s k^3}\hat{e}_u+\frac{1}{\tau_s k^2}\hat{e}_\mu-\frac{i}{\tau_s k^3}\hat{e}_\lambda \tag{87}$$

$$\hat{u}_\lambda\propto\frac{1}{\tau_s k^4}\hat{e}_u+\frac{i}{\tau_s k^3}\hat{e}_\mu+\frac{1}{k^2}\hat{e}_\lambda \tag{88}$$

For the magnitude of the Fourier coefficients we have,

$$|\hat{u}_k|\leq\frac{1}{k^2}|\hat{e}_u|+\frac{1}{\tau_s|k^3|}|\hat{e}_\mu|+\frac{|1-\tau_s|}{\tau_s k^4}|\hat{e}_\lambda|, \tag{89}$$

$$|\hat{u}_\mu|\leq\frac{1}{\tau_s|k^3|}|\hat{e}_u|+\frac{1}{\tau_s k^2}|\hat{e}_\mu|+\frac{1}{\tau_s|k^3|}|\hat{e}_\lambda|, \tag{90}$$

$$|\hat{u}_\lambda|\leq\frac{1}{\tau_s k^4}|\hat{e}_u|+\frac{1}{\tau_s|k^3|}|\hat{e}_\mu|+\frac{1}{k^2}|\hat{e}_\lambda|. \tag{91}$$

Which yields,

$$k^2|\hat{u}_k|\leq|\hat{e}_u|+\frac{1}{\tau_s|k|}|\hat{e}_\mu|+\frac{|1-\tau_s|}{\tau_s k^2}|\hat{e}_\lambda|, \tag{92}$$

$$k^2|\hat{u}_\mu|\leq\frac{1}{\tau_s|k|}|\hat{e}_u|+\frac{1}{\tau_s}|\hat{e}_\mu|+\frac{1}{\tau_s|k|}|\hat{e}_\lambda|, \tag{93}$$

$$k^2 \mid \hat{u}_\lambda \mid \leq \frac{1}{\tau_s k^2} \mid \hat{e}_u \mid + \frac{1}{\tau_s \mid k \mid} \mid \hat{e}_\mu \mid + \mid \hat{e}_\lambda \mid . \tag{94}$$

Assuming that k is sufficiently large so that $\mid k \mid > 1$, and $\mid k \mid > \frac{1}{\tau_s}$, we arrive at,

$$k^2 \mid \hat{u}_k \mid \leq \mid \hat{e}_u \mid + \mid \hat{e}_\mu \mid + \mid \hat{e}_\lambda \mid, \tag{95}$$

$$k^2 \mid \hat{u}_\mu \mid \leq \mid \hat{e}_u \mid + \frac{1}{\tau_s} \mid \hat{e}_\mu \mid + \mid \hat{e}_\lambda \mid, \tag{96}$$

$$k^2 \mid \hat{u}_\lambda \mid \leq \mid \hat{e}_u \mid + \mid \hat{e}_\mu \mid + \mid \hat{e}_\lambda \mid . \tag{97}$$

Noting that the left hand side represents the Fourier coefficient of $\frac{d^2 u_a}{dx^2}$, $\frac{d^2 \mu_a}{dx^2}$, and $\frac{d^2 \lambda_a}{dx^2}$, we conclude that,

$$\left\| \frac{d^2 u_a}{dx^2} \right\| \leq C_R \left(||e_u|| + ||e_\mu|| + ||e_\lambda|| \right) \tag{98}$$

$$\left\| \frac{d^2 \mu_a}{dx^2} \right\| \leq C_R \left(||e_u|| + \frac{||e_\mu||}{\tau_s} + ||e_\lambda|| \right) \tag{99}$$

$$\left\| \frac{d^2 \lambda_a}{dx^2} \right\| \leq C_R \left(||e_u|| + ||e_\mu|| + ||e_\lambda|| \right) \tag{100}$$

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