CHAPTER 12
MULTIPLE ELEMENTS THROUGH THE THICKNESS - ELEMENT CREATION

12.1 Point Creation

The introduction of the required number of elements in the mesh through the thickness is done by splitting edges of paths that have fewer edges than necessary (Figure 12.1). The edges of a path to be split are picked in decreasing order of their lengths. If there are fewer edges in the path than vertices to be added, edges may be split more than once. Edge split points are determined by bisection and subsequent snapping of the calculated point to the boundary, if the edge is a boundary edge. If snapping to the boundary results in the new regions being invalid, the edge is not split.

Figure 12.1: Creation of multiple nodes through the thickness by edge splitting - 2D example.
Since splitting of edges changes the paths determined in the initial opposite vertex search, the path information is updated as its edges are being split. This is more efficient than redoing the opposite vertex search.

12.2 Realignment of Edges

From Figure 12.1 it can be seen that splitting of path edges:

- does not eliminate all deficient paths through the mesh,
- creates new deficient paths through the thickness,
- creates large number of connections at some vertices, and
- results large face angles in 2D and large dihedral angles in 3D.

Therefore, realignment of the diagonal mesh edges along and perpendicular to the thickness direction is necessary as shown in Figure 12.1. The realignment of edges is accomplished through the use of edge swapping. The process of edge swapping is equivalent to retriangulation of the polyhedron formed by deletion of all regions connected to the edge such that the new triangulation does not contain any new points and does not contain the edge being swapped (Appendix A).

**Definition 12.1** A mesh edge, \( M^1_i \cap \overline{G^2_k} \), is defined as an opposite edge of another mesh edge, \( M^1_j \cap \overline{G^2_l} \), if each of the vertices of \( M^1_i \) are opposite to one of the vertices of \( M^1_j \).

**Definition 12.2** A mesh face, \( M^2_i \cap G^2_k \), is defined as an opposite face of another mesh face, \( M^2_j \cap G^2_l \), if each of the edges of \( M^2_i \) is opposite to one of the edges of \( M^2_j \).

The idea of opposite edges and faces is illustrated in Figure 12.2. In the figure, \( M^1_0, M^0_3, M^0_5, M^0_7 \) and \( M^0_9 \) are opposite to \( M^0_0, M^0_2, M^0_4, M^0_6 \) and \( M^0_8 \) respectively. Also, the edge \( M^1_1 \) is opposite to edge \( M^1_0 \) and the face \( M^2_1 \) is opposite to \( M^2_0 \). However, the edge \( M^1_2 \) connecting vertices \( M^0_4 \) and \( M^0_6 \) does not have an opposite edge since an edge does not exist between the opposite vertices, \( M^0_0 \) and \( M^0_7 \). Face
\(M_2^2\) does not have an opposite face because two of its edges do not have opposite edges. On the other hand, in spite of all vertices and edges of face \(M_3^2\) having opposite vertices and edges respectively, the face still does not have an opposite face since no common face connects the opposite edges.

![Illustration of opposite edges and faces.](image)

**Figure 12.2: Illustration of opposite edges and faces.**

### 12.2.1 Conversion of quads from diagonal to zigzag configurations

The knowledge of the special mesh topology created by splitting is used in the identification of mesh edges to swap and the sequence in which to swap them. The identification of edges to swap is facilitated by abstracting portions of the mesh between opposite faces as wedges (triangular prisms). The lateral faces of such wedges are abstracted as triangulated quadrilaterals. For example, shown in Figure 12.3 is a portion of a mesh in which multiple elements have been introduced through the thickness. In the figure, \(M_1^0\) is the opposite face of \(M_0^2\). One of the 3
abstracted quadrilaterals shown is formed by the vertex set $M_0^0, M_1^0, M_2^0, M_3^0$. The wedge shown is formed by the vertex set $M_0^0, M_1^0, M_2^0, M_3^0, M_4^0, M_5^0$.

Figure 12.3: Abstraction of mesh between opposite faces as a wedge.

The triangulation on a quadrilateral resulting from edge splitting is shown in Figure 12.4a(i) while the triangulation desired after realignment is depicted in Figure 12.4a(ii). The former triangulation is called a *diagonal triangulation* while the latter a *zigzag triangulation*. A wedge has a diagonal or a zigzag configuration if all of its quadrilateral faces have a diagonal (Figure 12.4b(i)) or a zigzag triangulation (Figure 12.4b(ii)) respectively. If some of the quadrilateral faces have a diagonal triangulation and others have a zigzag triangulation, the wedge is said to be partially zigzag (Figure 12.4b(iii)). Any other configuration is a general configuration that cannot be classified and is dealt with by general mesh modification procedures. Note that the sides of the quadrilaterals in the thickness direction are edge paths between opposite vertices.

With this abstraction defined, a major component of the process of realigning edges along and perpendicular to the thickness direction can be viewed as a conversion of all triangulated quadrilaterals in the mesh from a diagonal to a zigzag configuration. This allows the edge realignment process to be driven largely by topological considerations and is therefore more efficient than using geometric criteria.
The conversion of the diagonal triangulation on each quadrilateral to zigzag is done in a templated sequence of swaps illustrated in Figure 12.5. This sequence can be easily generalized for a quadrilateral with any number of edges through its thickness. Consider a diagonal quadrilateral with $N_v$ vertices along its lateral sides. Let the vertices of the two sides of the quadrilateral be labeled as \{${M_0^0, M_1^0, M_2^0, \ldots, M_{N_v-1}^0}$\} and \{${M_1^0, M_1^1, M_1^2, \ldots, M_{N_v}^0}$\}. Then the edge swapping sequence for converting a diagonal quad configuration to a zigzag one can be written as follows

\[
\text{for } i = 0 \text{ to } N_v - 1 \text{ do} \\
\qquad \text{for } j = N_v - 1 \text{ to } i + 2 \text{ do} \\
\qquad\qquad \text{Swap edge between } M_{0,i}^0, M_{1,j}^0 \text{ to edge between } M_{0,i+1}^0, M_{1,j-1}^0 \\
\qquad \text{end for} \\
\text{end for}
\]

If the triangulation created by splitting is such that a quadrilateral is not completely diagonal or completely zigzag then multiple iterations of the above of swaps
is applied to the edges of the quad so that it may be converted into a zigzag configuration. Before each swap it is checked whether the edge actually exists between the vertices $M_{0,i}$ and $M_{1,j}$.

If the wedge topology is not present in a portion of the mesh, it is still possible to introduce multiple elements through the thickness and realign mesh edges to eliminate deficient paths through the thickness. For general mesh topology through the thickness, edges may be realigned using geometric criteria such as the thickness direction in the local neighborhood. Also, more complex topology of the mesh requires general refinement methods. However, a small number of other cases which can be dealt with are handled specifically in the code and are described below.
12.2.2 Triangle and tetrahedral configuration

A special situation dealt with in the algorithm is two adjacent mesh vertices having the same opposite mesh vertices (Figure 12.6a). In this case, the edge connected to the opposite vertex in each path is ignored and the remaining edges used for the quadrilateral abstraction. Swapping of edges then proceeds as usual with this topology. Similarly, three vertices of a mesh face may have a common opposite vertex forming a master tetrahedron. The three edge paths are split and the edges are swapped on the faces of the master tetrahedron to eliminate any deficient paths in the tetrahedron.

12.2.3 Unswappable diagonal quad

If a diagonal quad’s edges cannot be swapped to convert it into a zigzag configuration, then an alternative approach is adopted to eliminate deficient paths in the quad. In this approach, the main diagonal of the quad is also split as many times as the sides of the quadrilateral (See Figure 12.6b). This creates two diagonal configuration “triangles” which can be converted to a zigzag configuration as described above (Figure 12.6b). Further, the direction of the diagonals of the two “triangles” is switched, if possible, since this leads to better face angles. It has been seen that this approach often succeeds when conversion by swapping alone fails.

![Figure 12.6: (a) Conversion of triangle from diagonal to zigzag configuration. (b) Conversion of quad into two zigzag triangles.](a) (b)

12.2.4 V-triangulation

In the V-quad configurations, a pair of vertices have opposite vertices but the edge between them does not have an opposite edge (or vice-versa). This is illustrated
in Figure 12.7a(i). In this case the diagonal edges comprising the V-configuration are split as many times as the lateral edges of the quad giving rise to three diagonal configuration quads which are then converted to zigzag (Figure 12.7a(ii-iv)).

### 12.2.5 Star configuration

In the star configuration, the main diagonal of a regular quad is split into two as shown in Figure 12.7b-(i). As before, the original diagonal edges of the quad are split to form a total of six diagonal configuration triangles (Figure 12.7b-(ii)). Thus the star configuration quad can be viewed as a V-quad and an inverted V-quad together. Conversion of the diagonal configuration “triangles” to zigzag follows the same procedure as before (Figure 12.7b(iii-iv)).

![Star configuration diagram](image)

Figure 12.7: Special quad configurations. (a) V-configuration and its conversion. (b) Inverted V-configuration and its conversion (c) Star configuration and its configuration.

The removal of deficiencies in a general configuration mesh using the procedures described up to this point is illustrated in Figure 12.8.
Figure 12.8: Elimination of deficiencies in a general mesh configuration. (a) Initial mesh. (b) Mesh after splitting. (c) Mesh after swapping.

12.3 Constraints in Reconfiguring Wedges using Local Mesh Modifications

The description of the realignment procedure until now assumed that multiple elements were introduced into a mesh which had only one element through the thickness. If the initial mesh had more than one element through the thickness and additional elements were introduced, then multiple wedges, stacked on top of each other, will exist between opposite faces. The procedures for recognition of quadrilaterals and wedges accounts for this. Once the individual wedges through the thickness have been identified, the swapping sequence can be applied to the quadrilateral faces of the wedge as before.

When converting a quadrilateral triangulation from diagonal to zigzag, care must be taken that wedges on either side of the quadrilateral are not forced into a configuration for which a tetrahedronization does not exist. The following rules
may be stated about the validity of wedge configurations with 2 elements through the thickness (See Figure 12.3):

- **Rule 1**: If the directions\(^{11}\) of triangulations of all 3 quadrilaterals of a wedge is the same, then the wedge cannot be tetrahedronized, regardless of the type of the triangulations.

- **Rule 2**: If 2 of the triangulations are diagonal, their directions must be opposite for the wedge to have a valid tetrahedronization. The direction of triangulation of the third quadrilateral is immaterial.

- **Rule 3**: If 2 of the triangulations are zigzag, at least their directions must be the same for the wedge to have a valid tetrahedronization. In addition, if the third triangulation is zigzag, its direction must not violate Rule 1; if it is diagonal, its direction is immaterial.

**Corollary**: If the number of edges through the thickness of the wedge is more than 2, then no valid tetrahedronization is possible with 1 diagonal and 2 zigzag triangulations (i.e. Rule 3 no longer holds). Rule 1 and Rule 2 are still valid.

The invalidity of all wedge configurations with two zigzag and one diagonal triangulation for wedges with more than two elements through their thickness places an important restriction on the mesh enrichment process. This restriction is that multiple elements must be introduced iteratively with edges through the thickness being split only once before the realignment procedure is applied to the modified mesh. Therefore, if an initial mesh has one element through the thickness everywhere and \(N_t\) elements have been requested through the thickness, the splitting and realignment process has to be performed \(\log_2 N_t\) times rounded off to the next integer. To lift this restriction, the local nature of the diagonal to zigzag conversion process must be sacrificed and propagated out into the mesh.

If a quadrilateral triangulation could not be converted to zigzag due to one of its adjoining wedges becoming invalid, it is revisited later to account for the possibility that other quadrilaterals in the local neighborhood may have been changed.

\(^{11}\)The direction of triangulation of a quadrilateral indicates which end of the diagonals in the triangulation are at a higher level. It is indicated schematically by an arrow on the base edge pointing towards the side of the quadrilateral with the higher end of the diagonal.
Figure 12.9: Wedge configurations with 2 elements through the thickness. (a) Valid wedge configurations. (b) Invalid wedge configurations.

to zigzag allowing for successful conversion. Still, in their current form, the realignment procedures may be prevented from converting all quadrilateral triangulations to zigzag by the invalidity of certain wedge configurations, even with two edges through the thickness.
12.3.1 Elimination of remaining deficient paths

When the topology of the mesh between thin sections is more general than that described before, it is necessary to use general refinement isotropic techniques to introduce multiple elements through the thickness. Several refinement techniques by means of edge bisection such as Rivara bisection, Bansch’s method and alternate bisection are reviewed in [15]. Many of these techniques have been described to be stable in three dimensions (the refined mesh quality is bounded from above or below by the initial mesh quality). Although most of these techniques over-refine due to the propagation of non-conformity, a careful selection of a refinement technique adapted from the above can be used to obtain isotropic refinement with sufficient elements through the thickness. It can be shown that an important component of obtaining a good quality mesh by these refinement methods is the ability to modify the initial surface triangulation.

12.3.2 Creation of multiple layers by local remeshing

While the above procedures to eliminate deficient paths using local mesh modifications do a good job of eliminating the deficient paths in the mesh, they suffer from the following shortcomings:

- They cannot deal well with portions of the mesh with general topology through the thickness, i.e., portions which do not have a wedge or quadrilateral structure.

- Even if wedge topology exists in portions of the mesh, they cannot guarantee conversion of all wedges from a diagonal to zigzag configuration.

- They are less efficient than direct creation of elements knowing the final topology of the mesh.

Of the above, the second shortcoming is very restrictive and directly prevents the procedures from achieving the goal of eliminating all deficient paths through the mesh. Even though the initial and final configurations are valid, it can be shown that a step-by-step conversion process by local mesh modifications cannot convert
all wedges from a diagonal to a zigzag configuration. This can be illustrated by a simple example shown in Figure 12.10. In the figure, the base triangulation for a set of four wedges is shown with the arrows representing the diagonal directions for the wedges. Also, “D” or “Z” next to the edge indicates that the quad growing on top of the edge is a diagonal or a zigzag configuration respectively. Assume that the triangulations of the four outer quads are constrained. It can be seen from Rule 2 above (Section 12.3) that all the wedges of the initial configuration are valid. Also, by Rule 1, the final configuration is valid. However, to go from the initial to the final configuration, the four interior quads must be made zigzag one after another. However, by Rule 3, the configuration will be invalid since the two zigzag quads of the wedge have opposite diagonal directions. Therefore, it can be seen that incremental conversion of the mesh from one configuration to the other is not always possible with the above procedures in the presence of constraints.

![Figure 12.10:](image)

Illustration that step-by-step modifications of wedge triangulations from diagonal to zigzag is not always possible.

Hence, it is proposed that multiple elements through the thickness be created by deleting the portion of the mesh that is deficient and creating an anisotropic mesh with sufficient number of elements through the thickness in its place. In the following discussion the conditions under which a deficient portion of a mesh can be deleted and replaced with a sufficiently enriched set of wedges is investigated.

Consider a set of edge connected triangles upon which wedges with 1 edge through the thickness and connected to each other along the quadrilateral faces are to be built. It is known that a wedge with 1 edge through the thickness and all its
diagonals in the same direction cannot be tetrahedronized without the introduction of any new points. Therefore, given such a configuration it must be determined if \textit{it is possible to find a combination of directions for the diagonals of the connected set of wedges such that all the wedges have a valid triangulation.}

As before if we think of the wedges in terms of the base triangulations and diagonal directions associated with them, the above question can rephrased as whether \textit{it is possible to find a combination of diagonal directions on the edges of the base triangular mesh representing the connected set of wedges such that all the triangles have a valid combination of diagonal directions.}

The answer to the above question is shown below to depend on:

- whether the direction of the diagonals on the quadrilaterals bounding the set of wedges (referred to henceforth as \textit{bounding quadrilaterals}) is constrained.

- whether the surface triangulation (upon which the wedges are to be constructed) can be altered or not.

If the edges of the triangles are assigned “diagonal directions” arbitrarily, then some of the triangles may end up with an invalid configuration. The methods for correcting these invalid configurations without altering the base triangular mesh are:

1. Flipping the “diagonal direction on an edge of the invalid triangle if does not make an adjacent triangles configuration invalid (Figure 12.11a). Lohner [40] has proposed an iterative scheme using this method but this method is not proved to guarantee correction of all triangles.

2. Propagating the invalid configuration through the mesh until another invalid configuration is reached at which point the diagonal direction for the common edge may be flipped. The flip makes both triangle configurations simultaneously valid (Figure 12.11b,c). The process of propagating an invalid configuration involves successively making a neighboring triangle configuration invalid to make the current one valid.
3. Propagating the invalid configuration through the mesh to the boundary of the set of triangles where the diagonal direction of a bounding edge may be flipped (if permitted) to make the triangle configuration valid (Figure 12.11c).

If the diagonal directions on the bounding edges are not constrained, then it is always possible to find a valid combination of directions on the edges of all the triangles under consideration. In fact, it is possible to prove that only one bounding edge diagonal direction of the connected set of triangles needs to be unconstrained to always get a valid combination of directions everywhere in the connected set. To see this, consider a set of connected triangles with a certain number of invalid configurations. Since this is a connected set of triangles, there must exist a path connecting any pair of triangles. Every pair of invalid configurations can be propagated towards each other and nullified as described in method 2 above. Therefore, if the set has an even number of invalid configurations, they must nullify each other out. If on the other hand, it has an odd number of invalid configurations, then after nullifying every pair of invalid configurations, one invalid triangle configuration remains. This invalidity can be propagated out to the boundary where it becomes necessary to flip the direction on one of the boundary edges of the triangle. Therefore, only one edge on the boundary of a set of connected triangles must have no constraints on its diagonal direction.

If all the bounding edges of the set of triangles is constrained, then it may not be possible to always obtain a valid set of triangles. This can be easily seen by considering a single triangle in an invalid configuration with all its edges constrained. The only way to obtain a valid configuration in such a case is to refine the surface triangulation in specific ways.

Consider a triangle with an invalid combination of diagonal directions associated with its edges (Figure 12.12a). This situation may be rectified by one of several ways shown in Figure 12.12. In Figure 12.12b, the face is split at the centroid and the new edges assigned diagonal directions appropriately so as to form 3 valid triangle configurations. This form of refinement is not the most desirable since it leads to large face angles on the surface and large dihedral angles in the volume mesh. In Figure 12.12c, the large angles have been bisected by bisecting the original edges of
Figure 12.11: Fixing invalid wedge configurations by edge swapping. (a) Fixing one invalid triangle by swapping the diagonal direction on an edge. (b) Fixing a pair of invalid triangles by swapping the diagonal direction of their common edge. (c) Fixing an invalid configuration by propagation of the triangle to the boundary.
the triangle. In Figure 12.12d, only one of the edges is bisected and the direction on one of the split edges is flipped. Note that the last two techniques make adjacent triangles non-conforming (having more than three vertices) which can be fixed by a bisection of the non-conforming triangles. If the adjacent triangle was invalid to begin with, it can be further split in the same way as the first to make it valid. On the other hand, if it was valid to start with, then regardless of the configuration, a direction for the bisection edge can be found such that the resulting two triangles will always be valid. Thus it can be seen that the described method of refinement is always guaranteed to generate a combination of diagonal directions that will all be valid.

If the quality of the surface triangulation is to be preserved, it is not sufficient to terminate the refinement process in the adjacent triangle arbitrarily by bisection. This is because bisecting an edge at the midpoint may result in creation of new poorly shaped elements and the refinement process is no longer stable (the angles are not bounded from below or above by the worst angle in the initial mesh). The process of alternate bisection or longest edge bisection may be applied wherein the refinement is propagated until all the triangles shapes are acceptable. Both methods tend to over-refine due to propagation of non-conformity and must be used only when all other methods for obtaining a valid set of diagonal directions on the edges have failed.