HW2: Prob 1

- problem statement: If $u = cy$ and $v = cx$

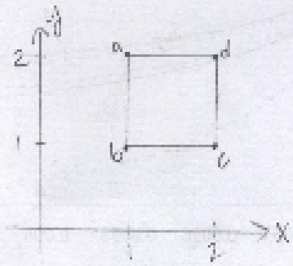
  a) Find the circulation about a box with corners (1,1) and (2,2)

  b) Find the potential ϕ if it exists.

- known: $u = cy$ $\quad v = cx$

- unknown: Γ, ϕ

- free body diagram:



- equations: $\Gamma = \oint \vec{V} \cdot ds = \int_a^b v\,dy + \int_b^c u\,dx + \int_c^d v\,dy + \int_d^a u\,dx$

  flow is irrotational if $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

  Then $\frac{\partial \phi}{\partial x} = u$ and $\frac{\partial \phi}{\partial y} = v$

- answers: $\Gamma = \int_2^1 c(1)\,dy + \int_1^2 c(1)\,dx + \int_1^2 c(2)\,dy + \int_2^1 c(2)\,dx$

  a)

  $= cy\big|_2^1 + cx\big|_1^2 + 2cy\big|_1^2 + 2cx\big|_2^1$

  $= -c + c + 2c - 2c = \boxed{0 \ \frac{m^2}{s}}$

  b) $\frac{\partial u}{\partial y} = c \qquad \frac{\partial v}{\partial x} = c \qquad \therefore$ flow is irrotational

  $\frac{\partial \phi}{\partial x} = cy \qquad \phi = cxy + f(y) + C_2$

$\frac{d\phi}{dy} = v = cx \qquad \phi = cxy + f(x) + C_3$

So $f(x) = f(y) = 0$ and $C_2 = C_3$. Then $\boxed{\phi = cxy + C_2}$

(2.7) (1)

- Problem statement: the velocity field in prob 2.3 is called source flow. For source flow, calculate:
  a) The time rate of change of the volume of a fluid element per unit volume.
  b) The vorticity
- Known: $u = \frac{cx}{x^2+y^2}$ $v = \frac{cy}{x^2+y^2}$
- Unknown: $\nabla \cdot \vec{V}$ and $\nabla \times \vec{V}$
- Equations: in polar coordinates

$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} \quad (1)$$

$$\nabla \times \vec{V} = \left[\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right]\hat{e}_z \quad (2)$$

- Answer

Transforming into the polar coordinates,

$$x = r\cos\theta \qquad y = r\sin\theta \quad (3)$$

$$V_r = u\cos\theta + v\sin\theta \qquad V_\theta = -u\sin\theta + v\cos\theta \quad (4)$$

$$u = \frac{cx}{x^2+y^2} = \frac{cr\cos\theta}{r^2} = \frac{c\cos\theta}{r} \quad (5)$$

$$v = \frac{cy}{x^2+y^2} = \frac{cr\sin\theta}{r^2} = \frac{c\sin\theta}{r} \quad (6)$$

substituting eqns (5) and (6) into eqns (4);

$$V_r = \frac{c}{r}\cos^2\theta + \frac{c}{r}\sin^2\theta = \frac{c}{r}$$

$$V_\theta = -\frac{c}{r}\cos\theta\sin\theta + \frac{c}{r}\cos\theta\sin\theta = 0$$

Now, calculating the divergence of $\vec{V}$ by using eqn (1) in polar coordinates;

a) $\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\frac{c}{\cancel{r}}\right) + \frac{1}{r}\overbrace{\frac{\partial V_\theta}{\partial \theta}}^{0} = \boxed{0}$

b) According to eq (2)

$$\nabla \times \vec{V} = \left[\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right]\hat{e}_z = [0+0-0]\hat{e}_z = \boxed{0}$$

Therefore, the flow is irrotational

2.9

- Problem statement: Is the flow field given in Prob 2.5 irrotational? Prove your answer
- Known $V_r = 0$ $V_\theta = cr$ c: constant
- **Unknown:** one way to verify if the flow is irrotational or not is calculating vorticity $\nabla \times \vec{V}$
- Equations: vorticity in cylindrical coordinates;

$$\nabla \times V = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix}$$

where $V_r = 0$
$V_\theta = cr$
$V_z = 0$

- Answer

$$\nabla \times V = \frac{1}{r}\frac{\partial}{\partial r}(cr^2)\,\hat{e}_z = \frac{1}{\cancel{r}}\,2c\cancel{r}\,\hat{e}_z = \boxed{2c\,\hat{e}_z}$$

The vorticity is finite and the flow must be <u>rotational</u>.