MANE4070: Aerodynamics I

Homework #8: Solution set

- Problem 5.1

→ Problem statement: consider a vortex filament of strength Γ in the shape of a closed circular loop of radius R. Obtain an expression for the velocity induced at the center of the loop in terms of Γ and R.

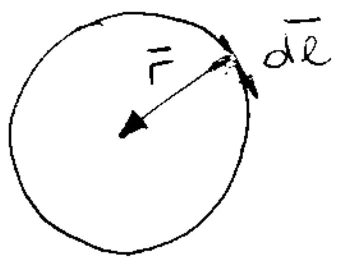
→ Known: R, Γ

→ Unknown: $\vec{V}$

→ Equations

$$\vec{V} = \frac{\Gamma}{4\pi}\int_{-\infty}^{\infty} \frac{d\bar{l} \times \bar{r}}{|r|^3}$$

→ Diagram



→ Solution

$d\bar{l} \times \bar{r} = R\,dl\,\bar{e}$ where $\bar{e}$ is a unit vector perpendicular to the plane of the loop, directed into the page.

$$\bar{V} = \frac{\Gamma}{4\pi R^2}\bar{e}\int dl = \frac{\Gamma}{4\pi R^2} 2\pi R\,\bar{e} = \frac{\Gamma}{2R}\bar{e}$$

Problem 5.2

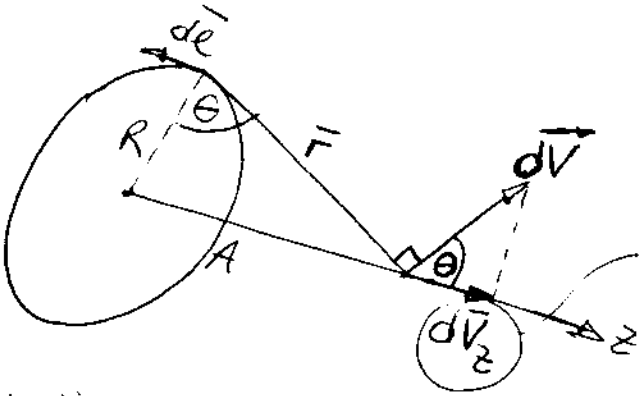
→ Problem statement: Consider the same vortex filament as in prob 5.1. Consider also a straight line through the center of the loop, perpendicular to the plane of the loop. Let A be the distance along this line. Obtain an expression for the velocity at distance A on the line

→ Known R, Γ, A

→ Unknown $\bar{V}$

→ Equations $\bar{V} = \int_{-\infty}^{\infty} \frac{\Gamma}{4\pi} \frac{\bar{dl} \times \bar{r}}{|\bar{r}|^3}$

→ Diagram



the problem requires this component

→ Solution

Since $\bar{dl}$ and $\bar{r}$ are always perpendicular:

$$d\bar{V} = \frac{\Gamma}{4\pi} \frac{\bar{dl} \times \bar{r}}{|\bar{r}|^3} = \frac{\Gamma}{4\pi} \frac{dl}{r^2} \qquad \text{where } \bar{dl} \times \bar{r} = |\bar{dl}|\,|\bar{r}| \overbrace{\sin 90^\circ}^{1}, \quad r^2 = R^2 + A^2$$

and $d\bar{V}_z = d\bar{V} \cos\theta$ where $\cos\theta = \frac{R}{\sqrt{R^2+A^2}}$

$$\bar{V}_z = \int d\bar{V}_z = \int d\bar{V} \cos\theta = \frac{\Gamma}{4\pi \underbrace{(R^2+A^2)}_{r^2}} \underbrace{\frac{R}{\sqrt{R^2+A^2}}}_{\cos\theta} \int_0^{2\pi R} dl$$

$$\Rightarrow \bar{V}_z = \frac{\Gamma R}{\cancel{4\pi}_{2} (R^2+A^2)^{3/2}} \cancel{2\pi} R = \boxed{\frac{\Gamma R^2}{2(R^2+A^2)^{3/2}}}$$