Error estimation and mesh optimisation using error in constitutive relation for electromagnetic field computation

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Abstract **– This paper presents a complete methodology to control the quality of electromagnetic field computation using the finite element method. An error estimate is built up using the error in constitutive relation. Proof is made that this estimate bounds up the exact error in some cases. Both problems of control of quality and mesh optimisation are then discussed.**

Introduction

The use of finite elements is now generalised in electromagnetic field computations. Now that efficient methods to solve field problems are available, it has become important to estimate the quality of a finite element solution and to control accuracy and computational costs. This problem is the one of error estimation.

Error estimation is based on the construction of an estimate which is a measure of the discretisation error. The construction of most of the error estimates is heuristic and gives only qualitative results [1-3]. The purpose of this paper is to present an estimate based on the error in constitutive relation which has the remarkable property to provide in certain cases an upper bound of the exact error (the word "exact" will be discussed). H-refinement technique [4] is used here to build up optimal meshes.

I. Error in Constitutive Relation

In the framework of electromagnetic problems, using a magnetic vector potential, the constitutive law is written $\mathbf{B}(\mathbf{A}) - \mu\mathbf{H} = 0$ where $\mathbf{B}$ is the magnetic flux density, $\mathbf{A}$ the magnetic vector potential as an unknown, $\mathbf{H}$ the magnetic field and μ the tensor of magnetic permeability (generally nonlinear).

When using a standard primal formulation, Ampere's law is only weakly satisfied [6]. Let assume now that a magnetic field $\mathbf{H'}$ which verifies exactly $\operatorname{curl}\mathbf{H'} = \mathbf{J}$ can be found a posteriori ($\mathbf{J}$ is the current density), then the couple $(\mathbf{B},\mathbf{H'})$ would be the exact solution of the problem only if $\mathbf{B} - \mu\mathbf{H'} = 0$ everywhere in the domain Ω. Otherwise, $(\mathbf{B},\mathbf{H'})$ is an approximation of the solution and the quantity $\mathbf{e} = \mathbf{B} - \mu\mathbf{H'}$ is a measure of the *error in constitutive relation* [4-5,7-8].

The global absolute error is defined as follow:

$$\mathbf{e}^2 = \int_\Omega (\mathbf{B} - \mu\mathbf{H'})^T \mu^{-1} (\mathbf{B} - \mu\mathbf{H'})\, d\Omega = \|\mathbf{B} - \mu\mathbf{H'}\|^2_{\mu^{-1},\Omega} \tag{1}$$

With the global absolute error $\mathbf{e}$ is associated a global relative error ε and with each element E is associated an elementary error ε_E as follows :

$$\varepsilon^2 \equiv \frac{\|\mathbf{B} - \mu\mathbf{H'}\|_{\mu^{-1},\Omega}}{\|\mathbf{B} + \mu\mathbf{H'}\|_{\mu^{-1},\Omega}} = \sum_E \frac{\|\mathbf{B} - \mu\mathbf{H'}\|_{\mu^{-1},E}}{\|\mathbf{B} + \mu\mathbf{H'}\|_{\mu^{-1},\Omega}} = \sum_E \varepsilon_E^2 \tag{2}$$

When using the error in constitutive relation, the exact error is in some cases an upper bound of the estimated error, i.e.

$$\varepsilon^2_{ex} \equiv \frac{\|\mathbf{B} - \mathbf{B}_{ex}\|_{\mu^{-1},\Omega}}{\|\mathbf{B} + \mathbf{B}_{ex}\|_{\mu^{-1},\Omega}} < \varepsilon^2 , \tag{3}$$

where the couple $(\mathbf{B}_{ex}, \mathbf{H}_{ex})$ is the exact solution of the problem. Starting from the identity

$$\mathbf{B} - \mu\mathbf{H'} = \mathbf{B} - \mathbf{B}_{ex} + \mu(\mathbf{H}_{ex} - \mathbf{H'}) , \tag{4}$$

and using (4) in (1), it comes:

$$\mathbf{e}^2 = \|\mathbf{B} - \mathbf{B}_{ex}\|^2_{\mu^{-1},\Omega} + \|\mathbf{H}_{ex} - \mathbf{H'}\|^2_{\mu,\Omega} + 2\int_\Omega (\mathbf{B} - \mathbf{B}_{ex})^T (\mathbf{H}_{ex} - \mathbf{H'})\, d\Omega \tag{5}$$

The orthogonality of the two affine subspaces containing $\mathbf{B}$ and $\mathbf{H'}$ respectively makes the third term of (5) vanish. Indeed, the integration by part of this third term gives:

$$\int_\Omega (\mathbf{B} - \mathbf{B}_{ex}).(\mathbf{H}_{ex} - \mathbf{H'})\, d\Omega = \int_\Omega (\mathbf{A} - \mathbf{A}_{ex}).\overbrace{\operatorname{curl}(\mathbf{H}_{ex} - \mathbf{H'})}^{0 \text{ in } \Omega}\, d\Omega + \int_\Sigma \overbrace{(\mathbf{H}_{ex} - \mathbf{H'}) \times \mathbf{n}}^{0 \text{ on } \Sigma_{\text{Neuman}}}.\overbrace{(\mathbf{A} - \mathbf{A}_{ex})}^{0 \text{ on } \Sigma_{\text{Dirichlet}}}\, d\Sigma \tag{6}$$

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This is only true if the Ampere law is satisfied exactly and if eddy currents are not taken into account (i.e. the current is known everywhere). In this case, the error in constitutive relation is an upper bound of the exact error.

Note that this method can also be applied to eddy current problems, then the estimate is no more an upper bound of the exact error. However, energy bounds cannot be established for eddy current formulations [6].

II. Construction of the Dual Field

A field **H'** can be built up only with local calculations if one uses the results of the primal formulation.

A. Calculation of projections P_{ij}

Let us consider the neighborhood of a node i (whose the associated nodal shape function is ω_i) composed of the elements adjacent to this node [E_j, j=1 to N] (Fig. 1).

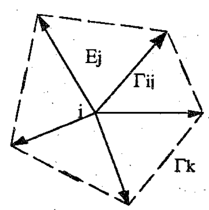

Fig. 1. The local problem defined by elements adjacent to node i.

Let us write the primal formulation for each element E_j:

$$\begin{aligned} P_{i,j+1} - P_{i,j} &\equiv \int_{\Gamma_{i,j+1}} \omega_i \mathbf{H'}\, dl - \int_{\Gamma_{i,j}} \omega_i \mathbf{H'}\, dl \\ &= -\frac{1}{2} \int_{E_j} \mu^{-1} \mathrm{curl}\, \mathbf{A}\, \mathrm{curl}\, \omega_i dE + \int_{E_j} \mathbf{J}\, \omega_i dE \end{aligned} \tag{7}$$

Note that boundary contributions are considered in each finite element wherein the magnetic field **H'** is a new unknown which is therefore not defined by μ^{-1}curl**A**.

The shape function associated with the node i vanishes on the boundary Γ_k (dashed edges). The calculation of the projections P_{ij} leads to the local problem:

$$\begin{cases} P_{i,2} - P_{i,1} = \mathbf{Y}_1(\mathbf{A}) \\ P_{i,3} - P_{i,2} = \mathbf{Y}_2(\mathbf{A}) \\ \quad \dots \\ P_{i,1} - P_{i,N} = \mathbf{Y}_N(\mathbf{A}) \end{cases} \tag{8}$$

System (8), where the $\mathbf{Y}_k$ are the right hand side of (7), is obviously underdetermined: the sum of the left hand sides of (8) is equal to zero, as well as the sum of the right hand sides, because the sum of these equations corresponds to the finite element equation associated with node i and it has been solved in the primal problem. The system becomes fully determined by replacing one equation of (8) by:

$$\sum_{k=1}^{N} P_{i,k} = \sum_{k=1}^{N} \int_{\Gamma_{i,k}} \omega_i \frac{\left(\mu^{-1}\,\mathrm{curl}\,\mathbf{A}\right)_{E_{k+1}} + \left(\mu^{-1}\,\mathrm{curl}\,\mathbf{A}\right)_{E_k}}{2} \cdot \mathbf{dl} \tag{9}$$

In (9), the difference between **H'** and μ^{-1}curl**A** is minimised with the least square criterion. The choice of (9) is arbitrary but it is easy to see that it makes the system regular.

B. Construction of H' knowing the projections P_{ij}

Let us consider the example of the first order triangle shown in Fig. 2.

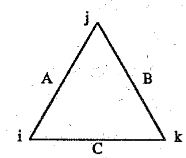

Fig. 2. First order triangle.

The local magnetic field $\mathbf{H}_E$ can be written as

$$\mathbf{H}_E = \sum_{\alpha=(A,B,C)} \left(h_{i,\alpha} s_{i,\alpha} + h_{j,\alpha} s_{j,\alpha}\right) \tag{10}$$

where i and j are the extremity nodes of the edge α, $h_{i,\alpha}$ is a scalar value for the connector associated with the node i of the edge α and the vector interpolating functions are constructed as follows :

$$\begin{cases} s_{iA} = \omega_i\, \mathrm{grad}\, \omega_j \\ s_{jA} = \omega_j\, \mathrm{grad}\, \omega_i \\ \quad \dots \\ s_{kC} = \omega_k\, \mathrm{grad}\, \omega_i \end{cases} \tag{11}$$

It is easy to see that each $s_{p,Q}$ is equal to zero on each edge that does not contain the node p and is orthogonal to each edge other than Q. For the edge A, the connectors $h_{i,A}$ and $h_{j,A}$ can be found by solving:

$$\begin{pmatrix} h_{i,A} \\ h_{j,A} \end{pmatrix} = \begin{pmatrix} \int_A s_{i,A}\, \omega_i\, \mathbf{dl} & \int_A s_{j,A}\, \omega_i\, \mathbf{dl} \\ \int_A s_{i,A}\, \omega_j\, \mathbf{dl} & \int_A s_{j,A}\, \omega_j\, \mathbf{dl} \end{pmatrix}^{-1} \begin{pmatrix} P_{i,A} \\ P_{j,A} \end{pmatrix} \tag{12}$$

and the same procedure is applied to the other edges B and C.

III. Mesh Optimisation

The optimal mesh M^* (with N^* elements and a global relative error ε^*) is built up by calculating a reduction size coefficient r_E for each element E of size h_E of the initial mesh M ($r_E = h^*_E/h_E$ where h^*_E is the optimal size of element E). The optimisation can be relative to either the minimization of N^* while keeping $\varepsilon^*<\varepsilon_0$ or the minimisation of ε^* while keeping $N^*<N_0$.

The efficiency θ of the estimate is defined as the ratio of the computed error ε to the exact error ε_{ex}. This coefficient is always higher than 1 when using the error in constitutive relation and when no eddy currents are present.

III. Numerical Examples

Two examples are considered. Geometries are discretised with first order triangular elements.

Example 1

Let us consider the problem of two concentric rings $R<r_1,r_2>$ and $R'<r_2,r_3>$ ($r_1<r_2<r_3$) made of a magnetic material of permeability μ and for which an analytical solution is known (Fig. 3). Neuman boundary conditions are imposed on $r = r_1$ and $r = r_3$. Current sources are distributed in the ring R' with the density $\mathbf{J} = j_m \sin(2\alpha)\,\mathbf{e}_z$.

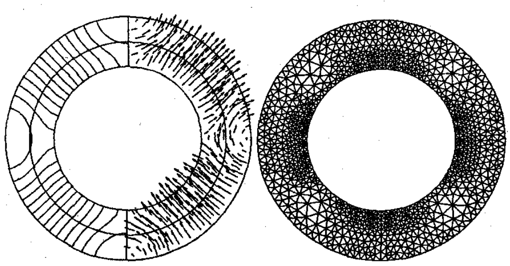

Fig. 3. Field lines, magnetic flux density and optimised mesh for $\varepsilon_0 = 3\%$.

Tables 1 and 2 present the results for two optimised meshes starting from a uniform initial mesh. In the first case, the objective was a 3% error (Fig. 3).

For the second case, the constraint is no more relative to the error ε but to the number of elements of the optimised mesh which was wished to be about 1890.

	N	ε_{ex} (%)	ε (%)	θ
initial mesh	486	4.92	7.09	1.44
optimised mesh	1890	2.18	2.99	1.37

Table 1. Results of the optimisation procedure for $\varepsilon_0 = 3\%$.

	N	ε_{ex} (%)	ε (%)	θ
initial mesh	486	4.92	7.09	1.44
optimised mesh	2108	2.00	3.01	1.49

Table 2. Results of the optimisation procedure for $N_0 = 1890$.

Example 2

The problem of a T-shaped magnetic conductor is considered. Fig. 4 shows eddy currents iso-lines produced by an exterior sinusoidal excitation (only the solid conductor is shown). The meshes associated with Fig. 4 are also shown in Fig. 4

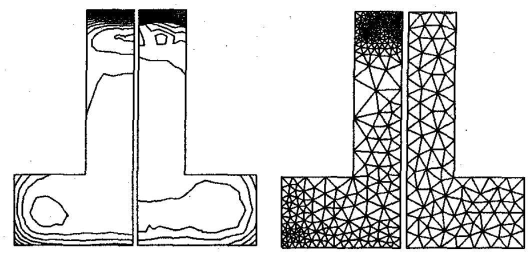

Fig. 4. From left to right : current lines for optimal mesh and for initial mesh, optimal and initial meshes.

The computed error varies from 25% for the initial mesh to 7% for the optimal mesh. Note that the value of the error does not constitute an upper bound of the exact error in this case, because of the presence of eddy currents.

Conclusions

An error estimate in constitutive relation has been applied successfully to electromagnetic field computation. The developed technique is actually valid for a wide range of problems. Its generalization to the computation of the error in transient and 3-D problems is the subject of further researches.

References

[1] N.A. Golias, T.D. Tsiboukis, "A-Posteriori adaptive mesh refinement in the finite element eddy current computation", *The International Journal for Computation and Mathematics in Electrical and Electronic Engineering, COMPEL*, Vol. 11, No 1, pp. 249-252, 1991.

[2] O.C. Zienkiewicz, J.Z. Zhu, "A simple error estimate and adaptive procedure for practical engineering analysis", *International Journal for Numerical Methods in Engineering*, Vol. 24, pp. 337-357, 1987.

[3] O.C. Zienkiewicz, J.Z. Zhu, "The superconvergent patch recovery and a posteriori error estimates: part I - The recovery technique", *International Journal for Numerical Methods in Engineering*, Vol. 33, pp. 1331-1364, 1992.

[4] P. Coorevits, *Maillage adaptatif anisotrope: application aux problèmes de dynamique*, PhD thesis, Ecole Normale Supérieure de Cachan, France, 1993.

[5] P. Ladveze, J.-P. Pelle, P. Rougeot,. "Error estimation and mesh optimisation for classical finite elements", *Engineering Computations*, Vol. 8, Pineridge Press, pp. 69-80, 1990.

[6] Chengjun LI, *Modélisation tridimensionnelle des systèmes électromagnétiques à l'aide de formulations duales/complémentaires. Application au maillage auto-adaptatif*, PhD Thesis, Université de Paris-Sud, France, 1993.

[7] P. Rougeot, *Sur le contrôle de la qualité des maillages éléments finis*, PhD thesis, University Paris 6, France, 1989.

[8] J.-P. Pelle, "Contrôle des Paramètres des Calculs Eléments Finis: Application au 3D et au Non-Linéaire", *Journée d'étude CSMA*, Paris, 1994.