Given $f_i: \Omega \to \mathbb{R}$, $g_i: \Gamma_{g_i} \to \mathbb{R}$ and $h_i: \Gamma_{h_i} \to \mathbb{R}$, find $u_i \in \delta_i$ such that

$$\int_\Omega w_{(i,j)} \sigma_{ij} \, d\Omega = \int_\Omega w_i f_i \, d\Omega + \sum_{\alpha=1}^{n_{sd}} \left(\int_{\Gamma_{h_i}} w_\alpha h_\alpha \, d\Gamma \right) \quad \forall w_i \in \mathcal{V}_i$$

(no sum in integrand)

with $\sigma_{ij} = C_{ijkl} \epsilon_{kl}$, $\epsilon_{kl} = u_{(i,j)} = \dfrac{u_{i,j} + u_{j,i}}{2}$

$$\delta_i = \{ u_i \mid u_i \in H^1, \; u_i = g_i \text{ on } \Gamma_{g_i} \} \quad i = 1(1) n_{sd}$$
$$\mathcal{V}_i = \{ w_i \mid w_i \in H^1, \; w_i = 0 \text{ on } \Gamma_{g_i} \} \quad i = 1(1) n_{sd}$$

1. Review text discussion of (S)→(W), (W)→(S)

note- $\int_\Omega w_{(i,j)} \sigma_{ij} \, d\Omega = \int_\Omega w_{(i,j)} C_{ijkl} \epsilon_{kl} \, d\Omega$

Stated with abstract form -

Given $\underset{\sim}{f}$, g and $\underset{\sim}{h}$ (defined over right stuff)
find $\underset{\sim}{u} \in \underset{\sim}{\delta}$ such that

$$a(\underset{\sim}{w}, \underset{\sim}{u}) = (\underset{\sim}{w}, \underset{\sim}{f}) + (\underset{\sim}{w}, \underset{\sim}{h})_\Gamma \quad \forall \underset{\sim}{w} \in \underset{\sim}{\mathcal{V}}$$

where

$$a(\underset{\sim}{w}, \underset{\sim}{u}) = \int_\Omega w_{(i,j)} C_{ijkl} \epsilon_{kl} \, d\Omega$$

$$(\underset{\sim}{w}, \underset{\sim}{f}) = \int_\Omega w_i f_i \, d\Omega$$

$$(\underset{\sim}{w}, \underset{\sim}{f})_\Gamma = \sum_{\alpha=1}^{n_{sd}} \int_{\Gamma_{h_\alpha}} w_\alpha h_\alpha \, d\Gamma \quad \text{(no sum.)}$$

We now work toward a "convenient" matrix description

we begin by representing the necessary terms from our second order tensors as terms listed in a vector-
(they still are second order tensors and transform coordinates like second order tensors - (not like vectors))

recalling that $\sigma_{ij} = \sigma_{ji}$

for $N_{sd} = 2$ $\sigma_{12} = \sigma_{21}$, so only 3 independent terms σ_{11}, σ_{22} and σ_{12} (or σ_{21})

Thus we will define

$n_{sd} = 2$ $\underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix}$

for $n_{sd} = 3$ we have $\sigma_{12} = \sigma_{21}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{23} = \sigma_{32}$
so $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}$ & σ_{23} can be the 6 independent terms
The text book ordering of the 6 terms

$n_{sd} = 3$ $\underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix}$

Likewise we define a 3×1 strain vector for $n_{sd}=2$ and 6×1 strain vector for 3D
How we convert from Green's strains to "Engineering strains" (double "shear" terms).

$$n_{sd}=2 \qquad \underset{\sim}{\epsilon}(u) = \{\epsilon_I(u)\} = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2}+u_{2,1} \end{Bmatrix}$$

$$n_{sd}=3 \qquad \underset{\sim}{\epsilon}(u) = \{\epsilon_I(u)\} = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{3,3} \\ u_{2,3}+u_{3,2} \\ u_{1,3}+u_{3,1} \\ u_{1,2}+u_{2,1} \end{Bmatrix}$$

We need our constitutive eq. in terms of the new stress and strain vectors

$$\underset{\sim}{T} = \underset{\sim}{D}\,\underset{\sim}{\epsilon}(u) \quad , \quad \underset{ex1}{\{\sigma\}} = \underset{exe}{[D]}\,\underset{ex1}{\{\epsilon(u)\}}$$

$[D]=[D]^T$
symmetric

$e = n_{sd}(n_{sd}+1)/2$,
$e=3$ for $n_{sd}=2$, $e=6$ for $n_{sd}=3$

If we are given the terms C_{ijke} using the known symmetries of C_{ijke} and $\underset{\sim}{D}$ as well as the relations from σ_{ij} to $\underset{\sim}{T}$ and ϵ_{ij} to $\underset{\sim}{\epsilon}(u)$ we can construct terms in $\underset{\sim}{D}$.
Text book shows how to do this
Since no one does this - I will not cover it or ask you to do it on homework or test.

with all this done we have

$$w_{(i,j)} C_{ijkl} u_{(k,l)} = \underset{\sim}{\epsilon}(\underset{\sim}{w})^T \underset{\sim}{\underset{\sim}{D}}\, \underset{\sim}{\epsilon}(\underset{\sim}{u})$$

$$a(\underset{\sim}{w},\underset{\sim}{u}) = \int_\Omega \underset{\sim}{\epsilon}(\underset{\sim}{w})^T \underset{\sim}{\underset{\sim}{D}}\, \underset{\sim}{\epsilon}(\underset{\sim}{u})\, d\Omega$$

Doing our normal decomposition of $\underset{\sim}{u}^h = \underset{\sim}{v}^h + \underset{\sim}{g}^h$, $\underset{\sim}{u}^h \in \underset{\sim}{\delta}^h$, $\underset{\sim}{v}^h \in \underset{\sim}{\mathcal{V}}^h$, $\underset{\sim}{g}^h \in \underset{\sim}{\delta}^h$

where:

$$\underset{\sim}{\mathcal{V}}^h = \{\underset{\sim}{w}^h \mid w_i^h \in \mathcal{V}_i^h\}, \quad \mathcal{V}_i^h = \{w_i^h \mid w_i^h \in H^1, w_i^h = 0 \text{ on } \Gamma_{g_i}\}$$

$$\underset{\sim}{\delta}^h = \{\underset{\sim}{u}^h \mid u_i^h \in \delta_i^h\}, \quad \delta_i^h = \{u_i^h \mid u_i^h \in H^1, u_i^h = g_i \text{ on } \Gamma_{g_i}\}$$

we have our Galerkin form

Given $\underset{\sim}{f}$, $\underset{\sim}{g}$ and $\underset{\sim}{h}$ as before find $\underset{\sim}{u}^h = \underset{\sim}{v}^h + \underset{\sim}{g}^h \in \underset{\sim}{\delta}^h$ such that

$$a(\underset{\sim}{w}^h, \underset{\sim}{v}^h) = (\underset{\sim}{w}^h, \underset{\sim}{f}) + (\underset{\sim}{w}^h, \underset{\sim}{h})_\Gamma - a(\underset{\sim}{w}^h, \underset{\sim}{g}^h)$$

To get to the matrix form we have to deal with the fact that essential BC can be associated with any nodes (in general dof holders)

(ndof ≥ nsd for vector systems)

Assuming nodal dof only where each node can have n_{dof}, lets use $n_{dof} = n_{sd}$ for the basic vector system

in this case the total possible number of dof is

$$n_{DOF} = n_{dof}\, n_{np} = n_{sd}\, n_{np}$$ for our current case

(n_{np} = # of nodes)

n_{eq} = actual # dof in global system

$n_{eq} < n_{DOF}$ $\Leftarrow$ must have sufficient essential B.C.

Consider again the set $\eta = \{1, 2, 3, \cdots, n_{np}\}$ and now define set

$$\eta_{g_i} = \{\text{nodes with a } g_i\}$$

with this we can write

$$v_i^h = \sum_{A \in \eta - \eta_{g_i}} N_A\, d_{iA} \qquad i = 1(1) n_{sd}$$

$$g_i = \sum_{A \in \eta_{g_i}} N_A\, g_{iA} \qquad i = 1(1) n_{sd}$$

Introducing yet another construct:
Euclidean basis vector $\underset{\sim}{e}_i$

$n_{sd} = 2$ $\quad \underset{\sim}{e}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$, $\underset{\sim}{e}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$

$n_{sd} = 3$ $\quad \underset{\sim}{e}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$, $\underset{\sim}{e}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$, $\underset{\sim}{e}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

with this we can write

$$\underset{\sim}{v}^h = v_i^h \underset{\sim}{e}_i \quad , \quad \underset{\sim}{g}^h = g_i^h \underset{\sim}{e}_i \qquad \text{(remember to sum)}$$

$$\underset{\sim}{w}^h = w_i^h \underset{\sim}{e}_i \quad , \quad w_i^h = \sum_{A \in \eta - \eta_{g_i}} N_A C_{iA}$$

Note - notation still a bit messy but idea is to include the correct terms in each summation as well as accounting for components -

Substituting all this in and recalling the C_{iA}'s are arbitrary we have

$$\sum_{j=1}^{n_{sd}} \left(\sum_{B \in \eta - \eta_{g_i}} a(N_A \underset{\sim}{e}_i, N_B \underset{\sim}{e}_j)\, d_{jB} = \right.$$

(Using the IDof = n_{sd} case)

$$(N_A \underset{\sim}{e}_i, \underset{\sim}{f}) + (N_A \underset{\sim}{e}_i, \underset{\sim}{h})_\Gamma$$

$$- \sum_{j=1}^{n_{sd}} \left(\sum_{B \in \eta_{g_i}} a(N_A \underset{\sim}{e}_i, N_B \underset{\sim}{e}_j)\, g_{jB} \right) \qquad \begin{matrix} A \in \eta - \eta_g \\ 1 \le i \le n_{sd} \end{matrix}$$

For writing the final matrix form we can use the ID device - now as a matrix

$$ID(i, A) \qquad \begin{matrix} i = 1(1)\, n_{sd} \\ A = 1(1)\, n_{np} \end{matrix} \qquad \text{(vector system nodal dof only)}$$

```
P=0
Do A= 1 to nnp
   Do i= 1 to nsd
      If A ∈ n - ngi
         P=P+1
         ID(i,A)=P   ← equation # in K
      else
         ID(i,A)=0
      end if
   end Do
end Do
neq=P ⇐ total number of actual equations
```

Using this we have

$$[K]\{d\} = \{F\} \qquad \left(\underset{\sim}{K}\,\underset{\sim}{d} = \underset{\sim}{F}\right)$$

$$P = ID(i,A) \;,\; Q = ID(j,B)$$

$$K_{PQ} = a(N_A \underset{\sim}{e}_i \,,\, N_B \underset{\sim}{e}_j)$$

$$F_P = (N_A \underset{\sim}{e}_i \,,\, \underset{\sim}{f}) + (N_A \underset{\sim}{e}_i \,\underset{\sim}{h})_\Gamma - \sum_{J=1}^{nsd}\left(\sum_{B \in n_{g_j}} a(N_A \underset{\sim}{e}_i \,,\, N_B \underset{\sim}{e}_j)\, g_{jB}\right)$$

⇒ We will see exactly how terms go into place when we look at our pseudo code later

To complete the ability to use vector matrix form we want to introduce the "B" matrix again

$$\underset{\sim}{\varepsilon}(N_A \underset{\sim}{e}_i) = \underset{\sim}{B}_A \underset{\sim}{e}_i \Leftarrow \text{Strain-displacement operator}$$

for $n_{sd}=2$ $\quad \underset{\sim}{B}_A = \begin{bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix}$

3×2 (3: # strain components; 2: n_{sd})

for $n_{sd}=3$

$$\begin{bmatrix} N_{A,1} & 0 & 0 \\ 0 & N_{A,2} & 0 \\ 0 & 0 & N_{A,3} \\ 0 & N_{A,3} & N_{A,2} \\ N_{A,3} & 0 & N_{A,1} \\ N_{A,2} & N_{A,1} & 0 \end{bmatrix}$$

6×3

With this we can write (not very useful!)

$$K_{PQ} = \underset{\sim}{e}_i^T \int_\Omega \underset{\sim}{B}_A^T \underset{\sim}{D} \underset{\sim}{B}_B \, d\Omega \, \underset{\sim}{e}_j$$

and noting that $\int_\Omega N_A \underset{\sim}{e}_i \underset{\sim}{f} \, d\Omega = \int_\Omega N_A f_i \, d\Omega$

$$\int_\Gamma N_A \underset{\sim}{e}_i \underset{\sim}{h} \, d\Gamma = \int_\Gamma N_A h_i \, d\Gamma$$

$$F_p = \int_{\Omega} N_A f_i \, d\Gamma + \int_{\Gamma_{h_i}} N_A h_i \, d\Gamma - \underbrace{\sum_{j=1}^{n_{sd}} \left(\sum_{B \in \eta_{g_j}} a(N_A \underset{\sim}{e_i}, N_B e_j) \, g_{jB} \right)}_{\text{still messy}}$$

Finally – We do not want to do those crazy integrals to directly get K_{PQ} and F_P terms

We want to do elements – the whole element – at once and have the assembly operator take care of putting things in the right place the right way

$$\underset{\sim}{K} = \overset{nel}{\underset{e=1}{A}} \underset{\sim}{k}^e \;, \quad \underset{\sim}{F} = \overset{nel}{\underset{e=1}{A}} (\underset{\sim}{k}^e, \underset{\sim}{f}^e)$$

n_{en} = # nodes / element

n_{ed} = # dof / node = n_{sd} for simple vector systems

$n_{ee} = n_{en} n_{ed}$ = # of dof of the element
→ will be the size of element $\underset{\sim}{k}^e, \underset{\sim}{f}^e$

$$\underset{\sim}{k}^e = [k^e_{pq}], \quad \underset{\sim}{f}^e = \{f^e\} \qquad 1 \le p,q \le n_{ee}$$

$$k_{pq} \; \underset{\sim}{e}_i^T \int_{\Omega_e} \underset{\sim}{B}_a^T \underset{\sim}{D} \underset{\sim}{B}_b \, d\Omega \, \underset{\sim}{e}_j \qquad \begin{aligned} p &= n_{ed}(a-1) + i \\ q &= n_{ed}(b-1) + j \end{aligned}$$

for $n_{sd} = 2$

$$\underset{\sim}{B}_a = \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \end{bmatrix}$$

for $n_{sd} = 3$

$$\underset{\sim}{B}_a = \begin{bmatrix} N_{a,1} & 0 & 0 \\ 0 & N_{a,2} & 0 \\ 0 & 0 & N_{a,3} \\ 0 & N_{a,3} & N_{a,2} \\ N_{a,3} & 0 & N_{a,1} \\ N_{a,2} & N_{a,1} & 0 \end{bmatrix}$$

what I called $\hat{f}_p^e$

$$f_p^e = \int_{\Omega^e} N_a f_i \, d\Omega + \int_{\Gamma_{hi}^e} N_a h_i \, d\Gamma$$

$$- \sum_{q=1}^{nee} k_{pq} g_q^e \qquad \Gamma_{hi}^e = \Gamma_{hi} \cap \Gamma^e$$

↑ portion of elements

$g_q = g_j(\underset{\sim}{x}_b^e)$ if node is on Γ_{g_j}
is zero otherwise

In actual programs we avoid the $\underset{\sim}{e}_i$'s and just do the whole thing at once for example –

Define $\underset{\sim}{B} = [\underset{\sim}{B}_1, \underset{\sim}{B}_2, \cdots \underset{\sim}{B}_{nen}]$

then $$\underset{\sim}{k}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D} \underset{\sim}{B} \, d\Omega$$

Stress evaluation

$$\underset{\sim}{\varepsilon}(u^h(x)) = \sum_{a=1}^{nen} \underset{\sim}{B}_a \underset{\sim}{d}_a^e \quad \text{dof at a node}$$

$$\underset{\sim}{\sigma}(x) = \underset{\sim}{D}(x) \underset{\sim}{\varepsilon}(u^h(x)) = \underset{\sim}{D}(x) \sum_{a=1}^{nen} \underset{\sim}{B}_a \underset{\sim}{d}_a^e$$

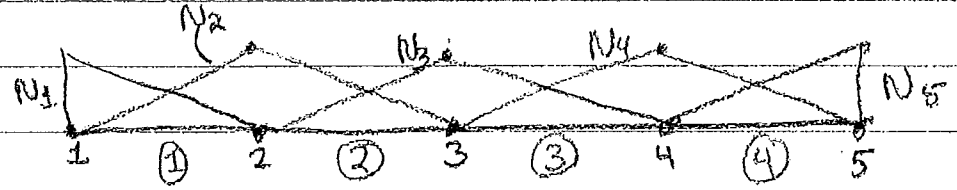
$$= \underset{\sim}{D}(x) \underset{\sim}{B} \underset{\sim}{d}^e$$

where

$$\underset{\sim}{B} = [\underset{\sim}{B}_1, \underset{\sim}{B}_2, \underset{\sim}{B}_3, \cdots \underset{\sim}{B}_{nen}], \quad \underset{\sim}{d} = \begin{Bmatrix} \underset{\sim}{d}_1^e \\ \underset{\sim}{d}_2^e \\ \vdots \\ \underset{\sim}{d}_{nen}^e \end{Bmatrix}$$

Global dof-sum element contributions

Lets Look at this for our specific case with 5 nodes

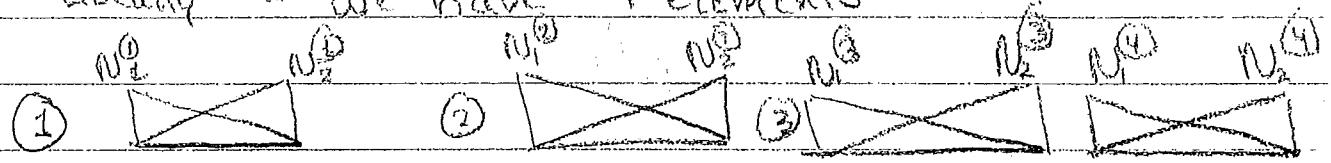


Global

$n_c = 4$, with essential bc on right need one more shape function for that → N_5

$\leftarrow N_{n+1}$

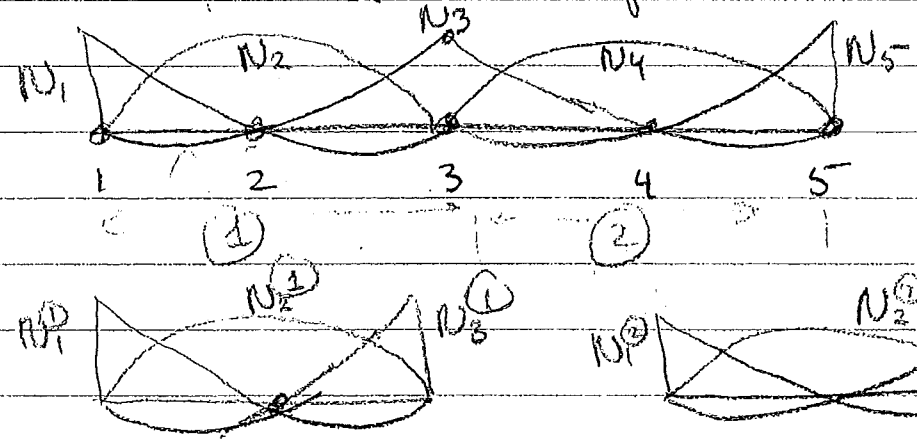
Locally ... we have 4 elements



$N_1 = N_1^{(1)}$, $N_2 = N_2^{(1)} + N_1^{(2)}$, $N_3 = N_2^{(2)} + N_3^{(1)}$

$N_4 = N_3^{(2)} + N_4^{(1)}$, $N_5 = N_2^{(4)}$

Can also do the quadratic ← Now 2-3 noded elements



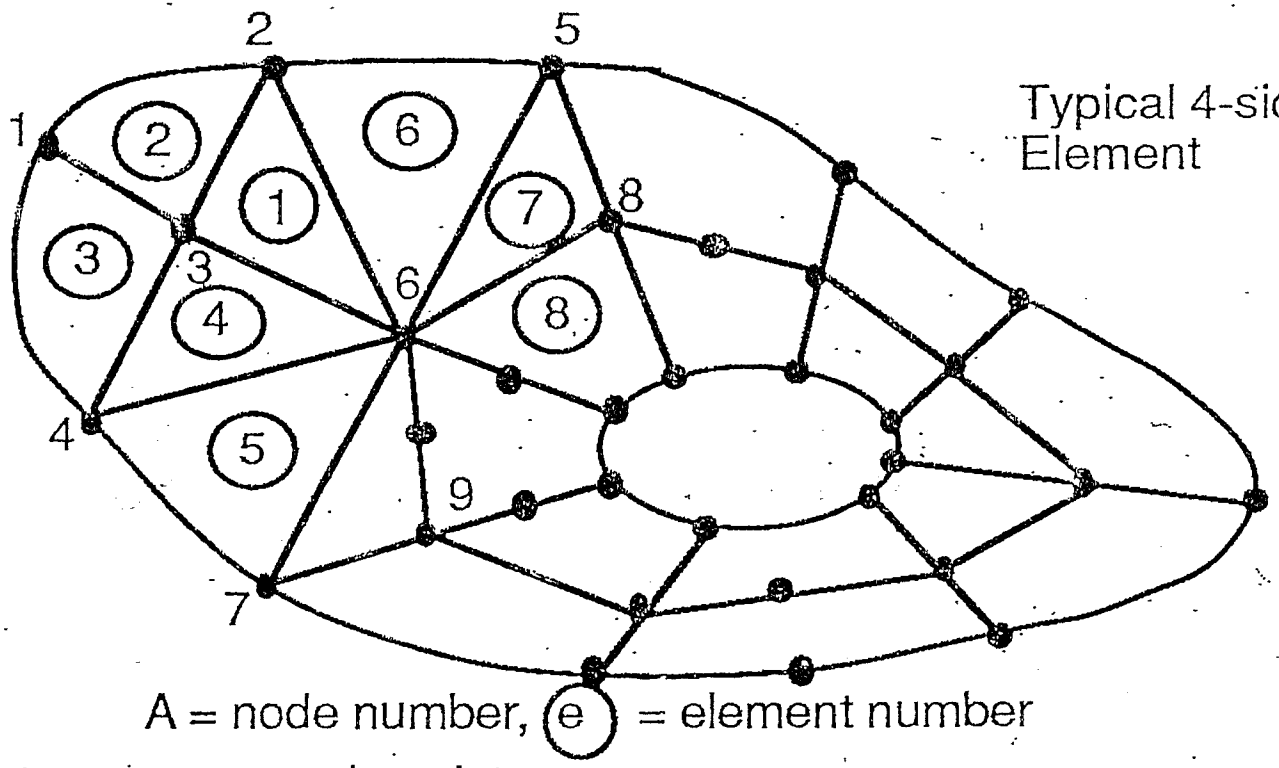
$N_e = 2$

$N_1 = N_1^{(1)}$, $N_2 = N_2^{(1)}$, $N_3 = N_3^{(1)} + N_1^{(2)}$,

$N_4 = N_2^{(2)}$, $N_5 = N_3^{(2)}$

Note - each shape function meets $N_A(x_B) = \delta_{AB}$

shape functions are continuous in value, but not slope between elements



A = node number, (e) = element number

• = node point

Typical 4-sided Element

Ω^e

$\Gamma^e = \Gamma^e_{int} + \Gamma^e_g + \Gamma^e_h$

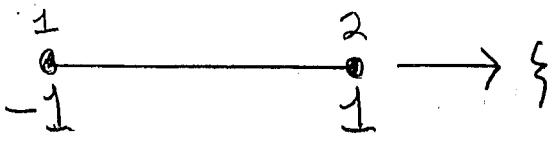
Ω^e

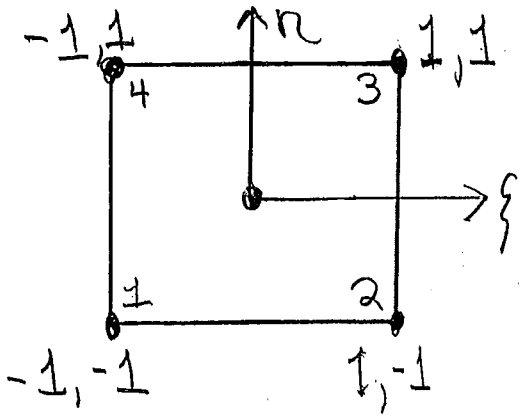
Typical 3-sided Element

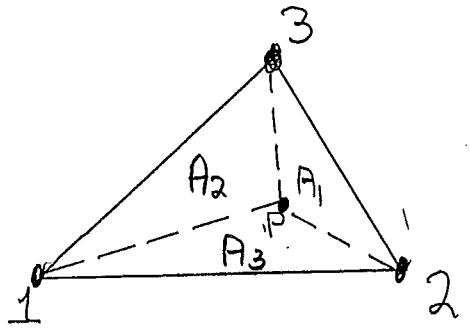
Local Coordinates -

It is not convenient to write our shape functions directly in the global coordinate system.

We prefer to write them in a local coordinate system and then deal with the mapping required







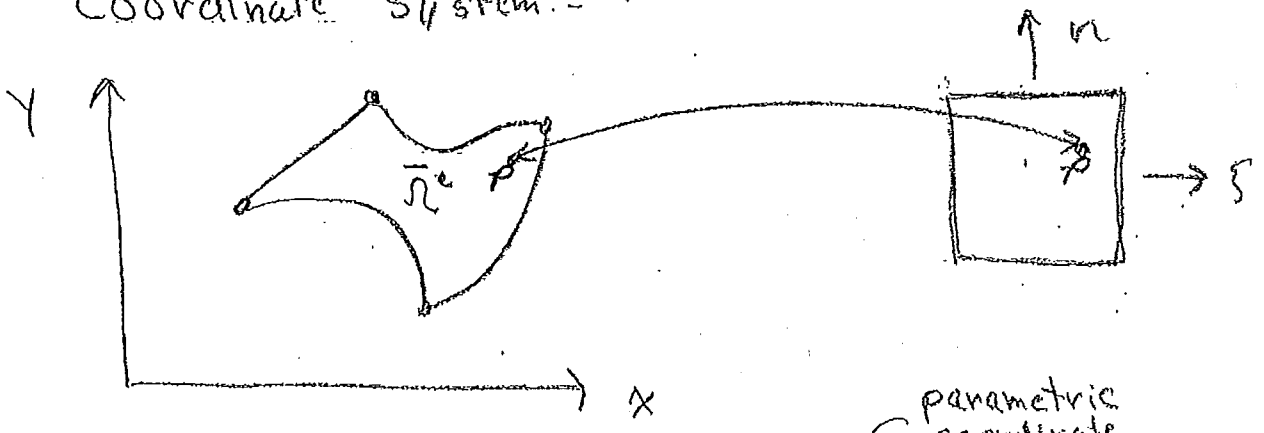
$$\xi_1 = A_1/A$$
$$\xi_2 = A_2/A$$
$$\xi_3 = A_3/A$$

Subject to: $\xi_1+\xi_2+\xi_3 = \frac{A_1+A_2+A_3}{A} = \frac{A}{A} = 1$

Use - $\xi_3 = 1-\xi_1-\xi_2$, $\xi_2 = 1-\xi_1-\xi_2$, or $\xi_1 = 1-\xi_2-\xi_3$

Lets revisit Mappings - a bit more general view

Relating our elements with shape functions defined in a parametric coordinate system with the element in the real (global) coordinate system -



parametric coordinate

need mapping acting on $\square$ to relate it to global coordinates -

going from parametric to real

$$\underset{\sim}{x} : \square \rightarrow \bar{\Omega}^e$$

We will assume elements are 1 to 1 ← To be discussed later

A useful mapping form would be -

$$\underset{\sim}{x} = \underset{\sim}{Q}^e(\xi, \eta)$$

- Common to define on a component by component basis -

$$x = Q^e_x(\xi, \eta) \quad ; \quad y = Q^e_y(\xi, \eta)$$

Now during the process of defining the stiffness matrix we need - $d\Omega$

$\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\int f(x,y)\,dx\,dy$

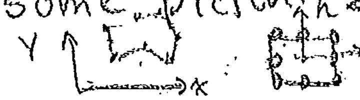
These are some basics – still need to have something to proceed –

Consider for now elements where we use nodal values of the components of $\underset{\sim}{u}$ as unknowns

$$\underset{\sim}{u}^e = \sum_{a=1}^{n_{en}} N_a \underset{\sim}{d}_a$$

$$u_i = \sum_{a=1}^{n_{en}} N_a d_{ia} \quad \text{where } d_{ia} = u_i \Big|_{\underset{\sim}{x} = \underset{\sim}{x}_a}$$

Could take the same type of approach – That is assume we define the set of geometry nodes for which we know the coordinates of the nodes in both $\underset{\sim}{x}$ and $\underset{\sim}{\xi}$ ← draw some pictures ← next page



define n^g_{nen} as number of geometric nodes

Then

$$\underset{\sim}{x} = \sum_{a=1}^{n^{\xi g}_{en}} {}^gN^g_a \underset{\sim}{x}^e_a \quad \text{where } \underset{\sim}{x}^e_a = \underset{\sim}{x}\Big|_{\text{node } a}$$

$$x_i = \sum_{a=1}^{n^g_{en}} {}^gN^g_a x^e_{ia}$$

The most common choise of ${}^gN_a$ is ${}^gN^g_a = N_a$ isoparametric

Define p^g as polynomial order of ${}^gN^g_a$

" p as polynomial order of N_a

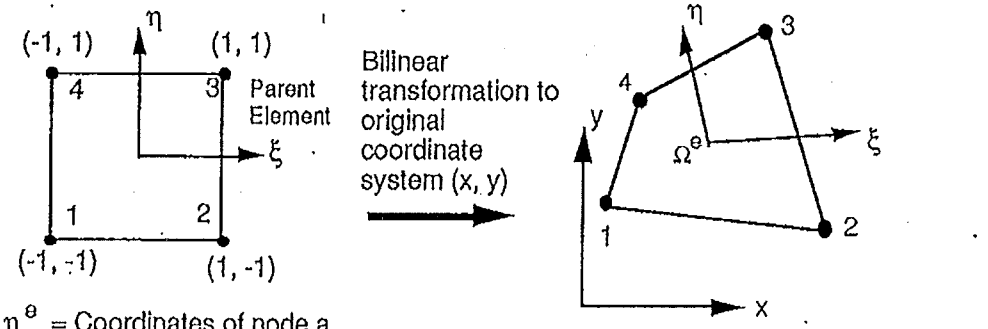
If $p^g > p$ superparametric element – does not exactly meet constant strain state part of C3

if $p^g < p$ subparametric – [illegible]

Isoparametric

- These interpolations map the square element in the parent ξ-η system into a more generally shaped element in x-y space whose shape depends on the interpolations, N_a, used.

For example, the linear (bilinear) interpolation maps the square into a quadrilateral with straight sides:

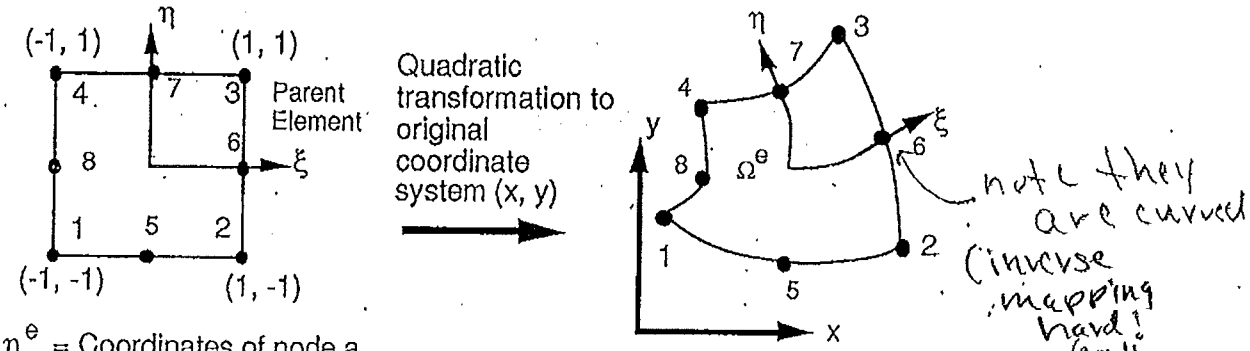


$\xi_a^e \eta_a^e$ = Coordinates of node a in the parent coordinate system

$x_a^e y_a^e$ = Coordinates of node a in the original coordinate system

d_a^e = Degree of freedom at node a

The quadratic (or biquadratic) interpolation maps the square into a quadrilateral with curved (quadratic) sides:



note they are curved (inverse mapping hard! Nonlinear

$\xi_a^e \eta_a^e$ = Coordinates of node a in the parent coordinate system

$x_a^e y_a^e$ = Coordinates of node a in the original coordinate system

d_a^e = Degree of freedom at node a

So is the bilinear but can be done

3-D elements add a third coordinate

note from before for iso parametric

$$\underset{\sim}{x} = \sum_{a=1}^{nen} N_a \underset{\sim}{x}_a$$ and by component.

$$x_i = \sum_{a=1}^{nen} N_a x_{ia}^e$$ or in x & y

$$x = \sum_{a=1}^{nen} N_a x_a^e$$

$$y = \sum_{a=1}^{nen} N_a y_a^e$$

Using these last two we can rewrite

$$\frac{\partial}{\partial x} = \frac{1}{|J|}\left(\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\right)$$

$$\frac{\partial}{\partial y} = \frac{1}{|J|}\left(-\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\right)$$

$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

Lets look at integration a bit closer consider 2-D

$$\int_{\Omega^e} f(x,y)\, d\Omega = \int_{-1}^{1}\int_{-1}^{1} f(x(\xi,\eta), y(\xi,\eta))\, |J(\xi,\eta)|\, d\xi\, d\eta$$

More specifically consider the element Stiffness matrix

$$\underset{\sim}{k}^e = \int_{\Omega^e} \underset{\sim}{B}^T \underset{\sim}{D}\, \underset{\sim}{B}\, d\Omega = \int_{a}^{b} N_{a,x}\, D\, N_{B,x}\, dx$$

$$\underset{\sim}{B} = [\underset{\sim}{B}_1, \underset{\sim}{B}_2 \cdots \underset{\sim}{B}_{nen}]$$

$N_a(\xi)$

for heat conduction

$$\underset{\sim}{B}_a = \underset{\sim}{\nabla} N_a = \begin{Bmatrix} N_{a,x} \\ N_{a,y} \end{Bmatrix} \qquad \underset{\sim}{D} = [k]_{2\times 2}$$

(assume its constant)

for elasticity

$$\underset{\sim}{B}_a = \begin{bmatrix} N_{a,x} & 0 \\ 0 & N_{a,y} \\ N_{a,y} & N_{a,x} \end{bmatrix} \qquad \underset{\sim}{D} = [D]_{3\times 3}$$

assume constant

So a typical term in the matrix is

$$\frac{\partial N_a}{\partial x} = \frac{1}{|J|}\left(\frac{\partial y}{\partial \eta}\frac{\partial N_a}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial N_a}{\partial \eta}\right) = \frac{1}{|J|}\left(y_{,\eta} N_{a,\xi} - y_{,\xi} N_{a,\eta}\right)$$

$$\frac{\partial N_a}{\partial y} = \frac{1}{|J|}\left(-\frac{\partial x}{\partial \eta}\frac{\partial N_a}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial N_a}{\partial \eta}\right) = \frac{1}{|J|}\left(-x_{,\eta} N_{a,\xi} + x_{,\xi} N_{a,\eta}\right)$$

substituting this in - gives

for heat conduction

$$\{\underset{\sim}{B}_a\} = \frac{1}{|J|}\begin{Bmatrix} Y_{,n}N_{a,s} - Y_{,s}N_{a,n} \\ -X_{,n}N_{a,s} + X_{,s}N_{a,n} \end{Bmatrix} = \frac{1}{|J|}\hat{\underset{\sim}{B}}_a$$

for elasticity

$$\underset{\sim}{B}_a = \frac{1}{|J|}\begin{bmatrix} Y_{,n}N_{a,s} - Y_{,s}N_{a,n} & 0 \\ 0 & -X_{,n}N_{a,s} + X_{,s}N_{a,n} \\ -X_{,n}N_{a,s} + X_{,s}N_{a,n} & Y_{,n}N_{a,s} - Y_{,s}N_{a,n} \end{bmatrix} = \frac{1}{|J|}\hat{\underset{\sim}{B}}_a$$

$$\underset{\sim}{B} = [\underset{\sim}{B}_1, \underset{\sim}{B}_2 \cdots \underset{\sim}{B}_{nen}] = \frac{1}{|J|}[\hat{\underset{\sim}{B}}_1, \hat{\underset{\sim}{B}}_2 \cdots \hat{\underset{\sim}{B}}_{nen}] = \frac{1}{|J|}\hat{\underset{\sim}{B}}$$

substituting this in yields the stiffness integral yields

$$\underset{\sim}{k}^e = \int \frac{1}{|J|}\hat{\underset{\sim}{B}}^T \underset{\sim}{D} \frac{1}{|J|}\hat{\underset{\sim}{B}} |J| \, ds \, dn$$

see why $|J| \neq 0$

$$k^e = \int_{-1}^{1}\int_{-1}^{1} \hat{\underset{\sim}{B}}^T \underset{\sim}{D} \hat{\underset{\sim}{B}} \frac{1}{|J|} ds \, dn \Leftarrow$$ in general this is a rational function

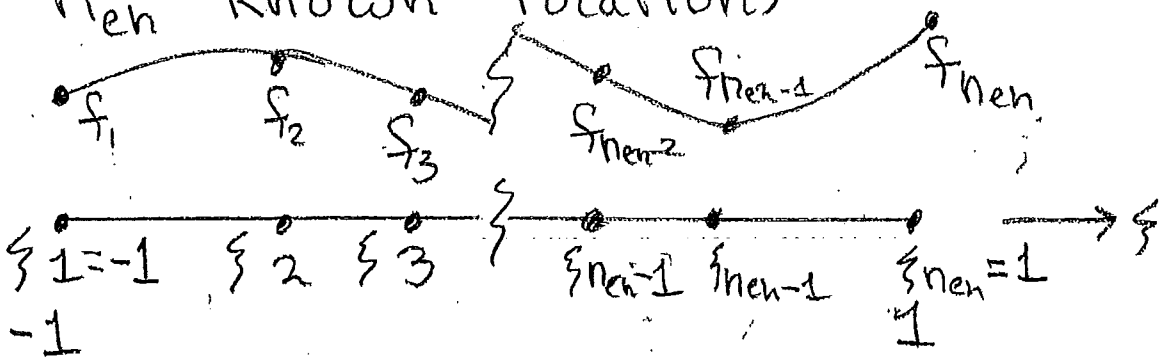
↳ when the element shapes are general since $|J|$ is in that case $f(\xi, n)$ when N_a has terms greater than linear

- when it is a rational function to utilize integration is no longer given by simple formula for polynomial integrate approximately

Lagrange Interpolation

A procedure to construct an order $n_{en}-1$ polynomial that interpolates a 1D function at n_{en} points

Starting point: knowledge of the value of the function at n_{en} known locations

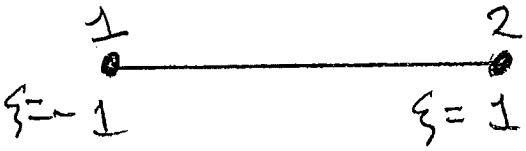


Lagrange Polynomials: An order $n_{en}-1$ order polynomial that interpolates at the n_{en} given locations ← Order of

$$f_i(\xi) = \sum_{a=1}^{n_{en}} l_a^{n_{en}-1}(\xi)\, f(\xi_a) \quad , \text{note } l_a^{n_{en}-1}(\xi_b) = \delta_{ab}$$

$$l_a^{n_{en}-1}(\xi) = \frac{\prod_{\substack{b=1 \\ b\neq a}}^{n_{en}} (\xi - \xi_b)}{\prod_{\substack{b=1 \\ b\neq a}}^{n_{en}} (\xi_a - \xi_b)}$$

Two node linear



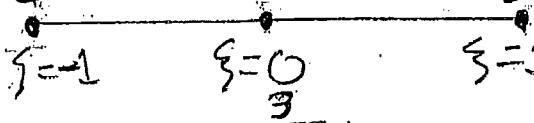
$n_{en}=2$

$n_{en}-1=1 \Leftarrow$ linear functions

$$l_1^1(\xi)=\frac{\prod_{\substack{b=1\\ b\neq 1}}^{2}(\xi-\xi_b)}{\prod_{\substack{b=1\\ b\neq 1}}^{2}(\xi_1-\xi_b)}=\frac{\xi-\xi_2}{\xi_1-\xi_2}=\frac{\xi-1}{-1-1}=\frac{1}{2}(\xi-1)$$

$$l_2^1(\xi)=\frac{\prod_{\substack{b=1\\ b\neq 2}}^{2}(\xi-\xi_b)}{\prod_{\substack{b=1\\ b\neq 2}}^{2}(\xi_2-\xi_b)}=\frac{\xi-\xi_1}{\xi_2-\xi_1}=\frac{\xi-(-1)}{1-(-1)}=\frac{1}{2}(\xi+1)$$

Three Node Quadratic (equially spaced)



$n_{en}=3$, $n_{en}-1=2$

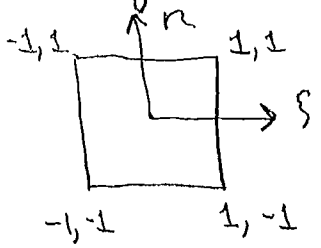
$$l_1^2(\xi)=\frac{\prod_{\substack{b=1\\ b\neq 1}}^{3}(\xi-\xi_b)}{\prod_{\substack{b=1\\ b\neq 1}}^{3}(\xi_1-\xi_b)}=\frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)}=\frac{\xi(\xi-1)}{(-1)(-2)}=\frac{1}{2}\xi(\xi-1)$$

$$l_2^2(\xi)=\frac{\prod_{\substack{b=1\\ b\neq 2}}^{3}(\xi-\xi_b)}{\prod_{\substack{b=1\\ b\neq 2}}^{3}(\xi_2-\xi_b)}=\frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)}=\frac{(\xi-(-1))(\xi-1)}{(1)(-1)}=1-\xi^2$$

$$l_3^2(\xi)=\frac{\prod_{\substack{b=1\\ b\neq 3}}^{3}(\xi-\xi_b)}{\prod_{\substack{b=1\\ b\neq 3}}^{3}(\xi_3-\xi_b)}=\frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)}=\frac{(\xi-(-1))(\xi-0)}{(1-(-1))(1-0)}=\frac{1}{2}\xi(\xi+1)$$

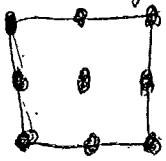
Constructions of 2-D - 4 sided elements Using Lagrange Shape functions.

As indicated previously, the element's shape functions will be written in the parametric coordinate system - Therefore, all elements are bi-unit squares



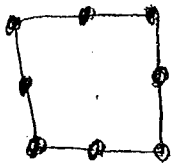
We will consider two families of such elements

(Full) Lagrangian



Will in general have interior and boundary nodes

Serendipity



- Nodes only on edges

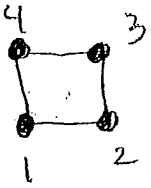
Also

Variable # nodes

Start with Lagrangian -

Elements defined by multiplying 1-D Lagrange polynomials in one direction times those in the other direction(s) 3-D

Start with simple 4-noded element

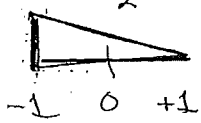


With 2 nodes per side we want to use linear

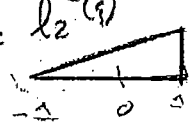
Consider the 1-D Lagrange Polynomials

In general - Note - 2 nodes per edge - linear

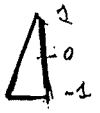
in ξ direction - $l_1^1 = \frac{1}{2}(1-\xi) = l_1^1(\xi)$



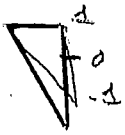
$l_2^1 = \frac{1}{2}(1+\xi) = l_2^1(\xi)$



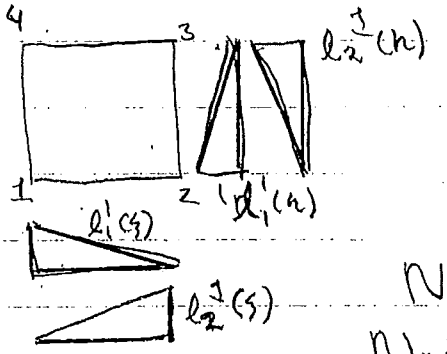
in η direction - $l_1^1 = \frac{1}{2}(1-\eta) = l_1^1(\eta)$



$l_2^2 = \frac{1}{2}(1+\eta) = l_2^2(\eta)$



the shape functions at the nodes are the "correct multiplication of these



$$u^h = \sum_{a=1}^{4} N_a d_a$$

$N_1 = l_1^1(\xi)\, l_1^1(\eta) = \frac{1}{4}(1-\xi)(1-\eta)$

$N_2 = l_2^1(\xi)\, l_1^1(\eta) = \frac{1}{4}(1+\xi)(1-\eta)$

$N_3 = l_2^1(\xi)\, l_2^1(\eta) = \frac{1}{4}(1+\xi)(1+\eta)$

$N_4 = l_1^1(\xi)\, l_2^1(\eta) = \frac{1}{4}(1-\xi)(1+\eta)$

Note - $N_a = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta) = \frac{1}{4}(1+\xi_a\xi+\eta_a\eta+\xi_a\eta_a\xi\eta)$

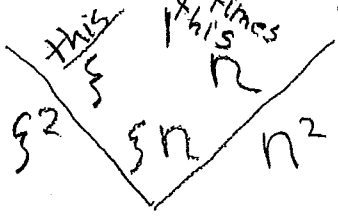
in other words each - $a_{11} = 1$

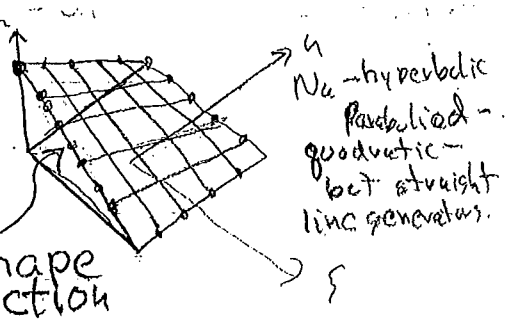
$N_a = a_{0a} + a_{1a}\xi + a_{2a}\eta + a_{3a}\xi\eta$

Since each shape function has same form

$u^h = a_0 + a_1\xi + a_2\eta + a_3\xi\eta$

Pascals triangle (in ξ-η)





Higher Order Lagrange elements

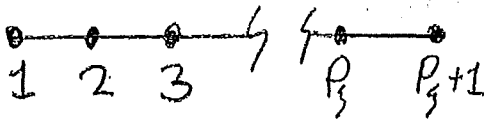
Products of 1-D shape functions

2D $N_a(\xi,\eta) = l_{a_\xi}^{P_\xi}(\xi)\, l_{a_\eta}^{P_\eta}(\eta)$

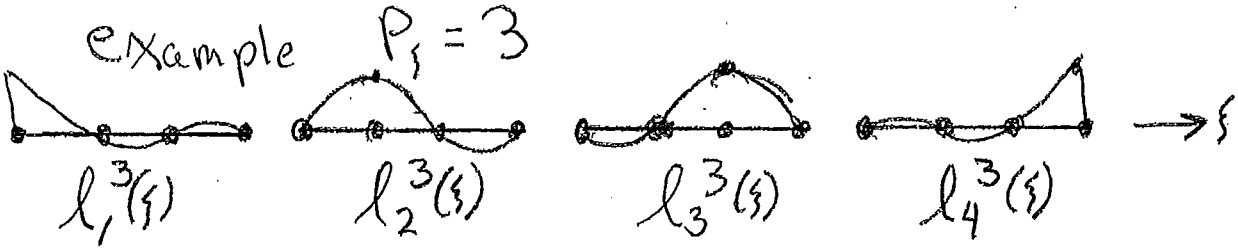
3D $N_a(\xi,\eta,\zeta) = l_{a_\xi}^{P_\xi}(\xi)\, l_{a_\eta}^{P_\eta}(\eta)\, l_{a_\zeta}^{P_\zeta}(\zeta)$

P_ξ = polynomial order in ξ direction

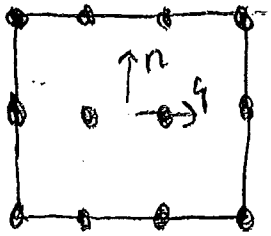
a_ξ - ξ direction "station" for node a



(Same for other coordinates

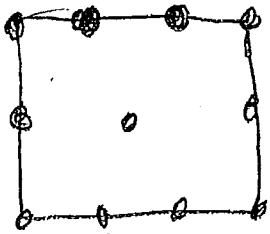


Use of this formula requires the same order polynomial in a specific direction at each station in the other directions



Cubic in ξ
quadratic in η

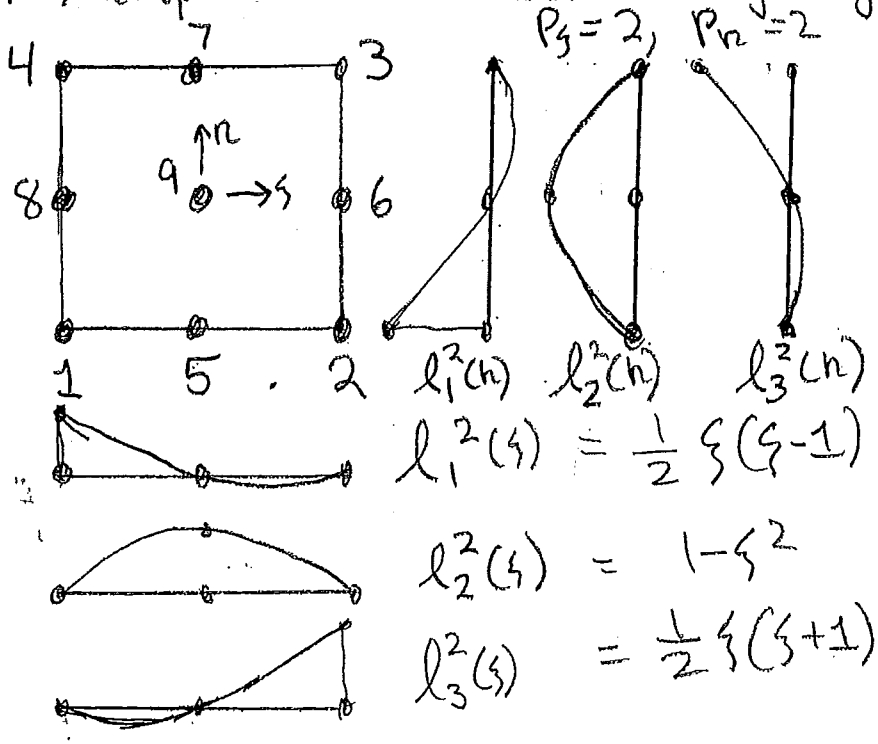
Lagrangian Element



← not cubic in ξ

Not a Lagrangian Element

Example - 9 noded Lagrangian Quadratic



$$l_1^2(\eta) = \tfrac{1}{2}\eta(\eta-1)$$
$$l_2^2(\eta) = 1-\eta^2$$
$$l_3^2(\eta) = \tfrac{1}{2}\eta(\eta+1)$$

$$l_1^2(\xi) = \tfrac{1}{2}\xi(\xi-1)$$
$$l_2^2(\xi) = 1-\xi^2$$
$$l_3^2(\xi) = \tfrac{1}{2}\xi(\xi+1)$$

$$N_1 = l_1^2(\xi)\, l_1^2(\eta) \qquad N_4 = l_1^2(\xi)\, l_3^2(\eta) \qquad N_7 = l_2^2(\xi)\, l_3^2(\eta)$$
$$N_2 = l_3^2(\xi)\, l_1^2(\eta) \qquad N_5 = l_2^2(\xi)\, l_1^2(\eta) \qquad N_8 = l_1^2(\xi)\, l_2^2(\eta)$$
$$N_3 = l_3^2(\xi)\, l_3^2(\eta) \qquad N_6 = l_3^2(\xi)\, l_2^2(\eta) \qquad N_9 = l_2^2(\xi)\, l_2^2(\eta)$$

Conditions C1-C3 ($m=1$ case)

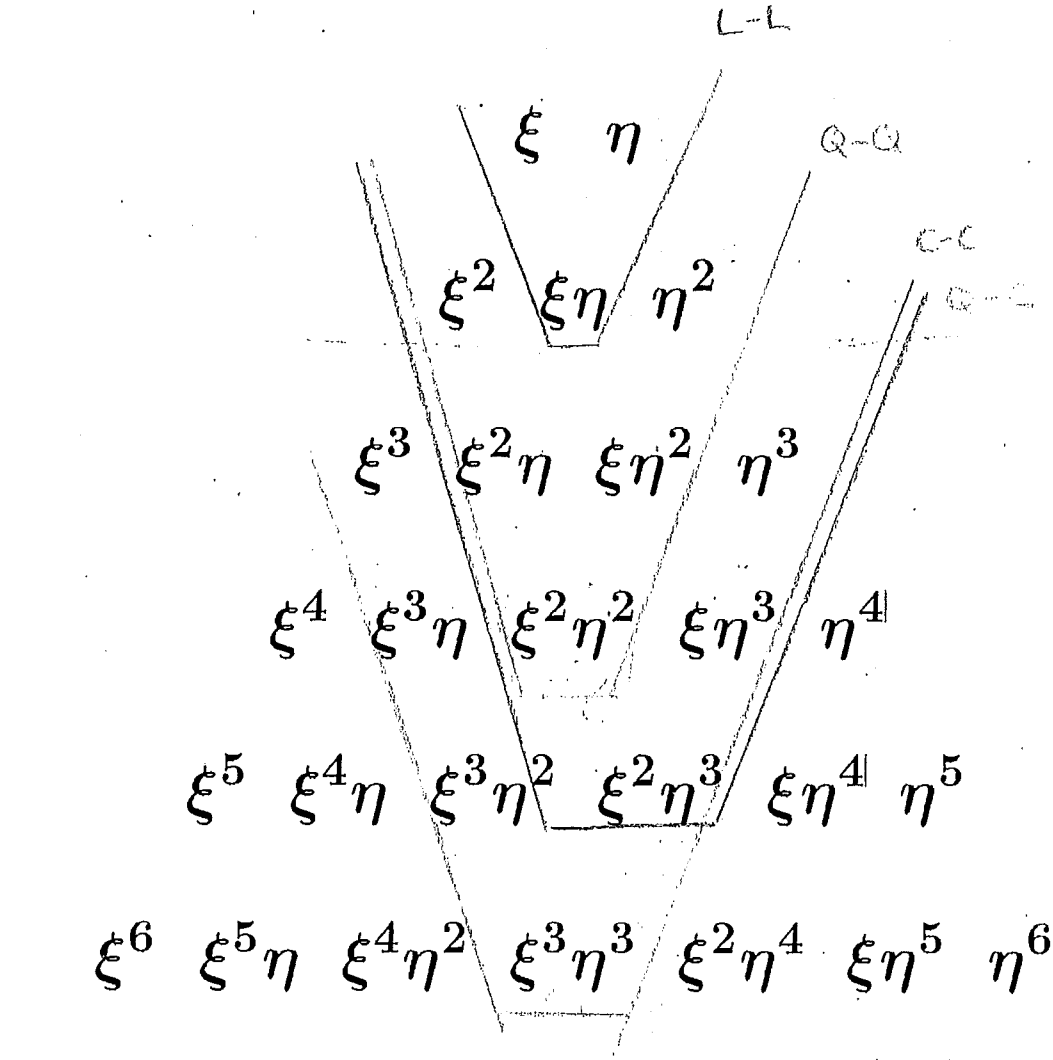
C1 - Intraelement continuity need C^1

In ξ-η, its fine | In x-y also need positive $|J|$ in mapping - more later

C2 - Inter element continuity - need C^0

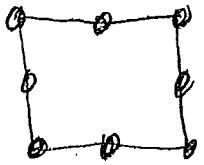
Need to show u^h is continuous between elements - Will again look at u on boundary between elements - If uniquely defined in terms of nodal values on the boundary - u^h is continuous since the nodal values are shared.

Lagrangian Elements

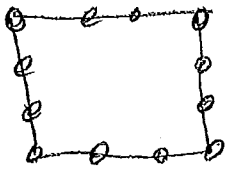


Serendipity element, variable # of nodes per edge, and other options

Serendipity - typically nodes on edges only

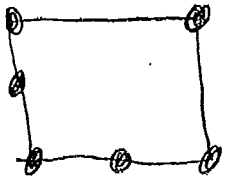


8 noded most common



12 noded

Variable # of nodes -



Useful for transition of element orders

Can not just apply products of Lagrange polynomials since those will have "zeros" that will not be accounted for - not enough conditions.

Ad-hoc construction methods required

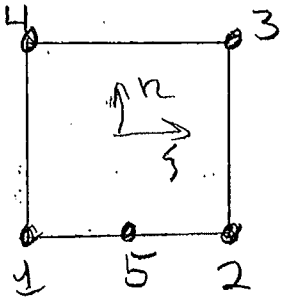
Typical procedure

1. Start with "minimal" Lagrangian element - typically will be bi-linear
2. Add proper shape functions for added nodes
3. Correct any previous shape functions to satisfy the $N_a(\xi_b) = \delta_{ab}$ condition.

easiest to simply work through examples.

$\hat{N}_a$ - starting "minimal" shape functions
N_a - the shape function when done

Start with easest example = an extra node



$\hat{N}_a$, a=1-4 The bi-linears.

Step 1 $\hat{N}_a = \frac{1}{4}(1+\xi_a\xi)(1+\eta_a\eta)$ a=1(1)4

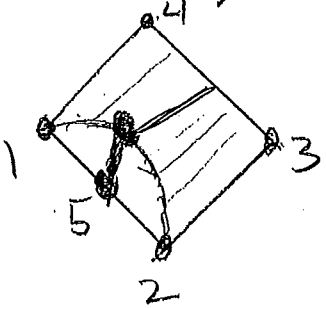
Step 2 - Define a shape function for node 5 that satisfies $N_a(\xi_b,\eta_b) = \delta_{ab}$

need function that is:

1 at node 5, 0 at nodes 1-4

- to be 1 at node 5 and 0 at nodes 1 and 2 need a quadratic in ξ. Note that $N_5|_{3-4} = 0$ is required so in the η direction

need $N_5(\xi,-1) = 1$ ↙ or something , $N_5(\xi,1) = 0$



$N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$

$l_2(\xi)$ $l_1'(\eta)$

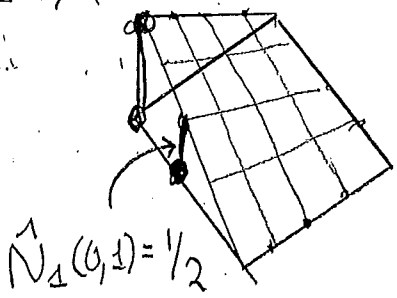
Step 3 correct any of the $\hat{N}_a$ shape functions that are not zero at the nodes of shape functions defined in step 2

Look at $\hat{N}_a(\xi_5, \eta_5)$ $a = 1(1)4$

$\hat{N}_a(0,-1)$ $a = 1(1)4$:

$\hat{N}_1(0,1) = 1/2$

$\hat{N}_a(0,1) = 0$, $a = 2(1)4$



$\hat{N}_1(0,1) = 1/2$

Easy to correct

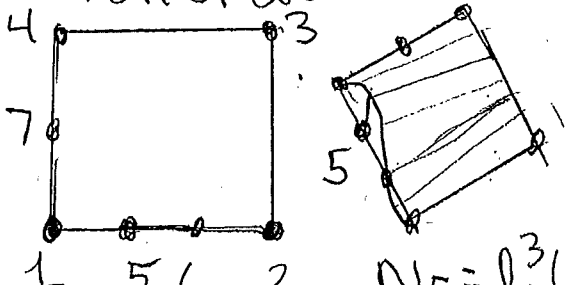
$$N_1 = \hat{N}_1 - \hat{N}_1(0,-1)\, N_5 = \hat{N}_1 - \tfrac{1}{2} N_5$$

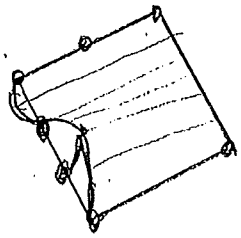
because $N_5 = 0$ at others - causes no problem

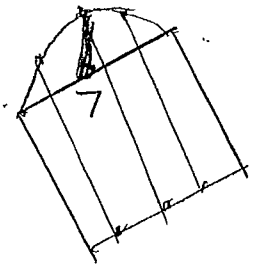
we end up with

$N_1 = \hat{N}_1 - \frac{1}{2}N_5$, $N_2 = \hat{N}_2 + \frac{1}{2}N_5$, $N_3 = \hat{N}_3$, $N_4 = \hat{N}_4$

Consider -



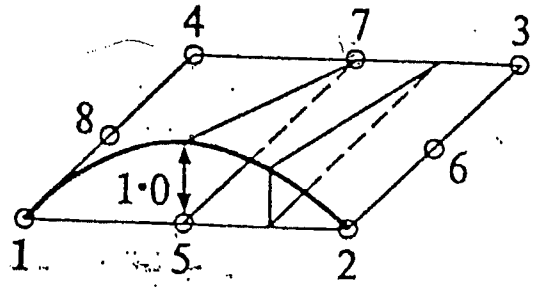


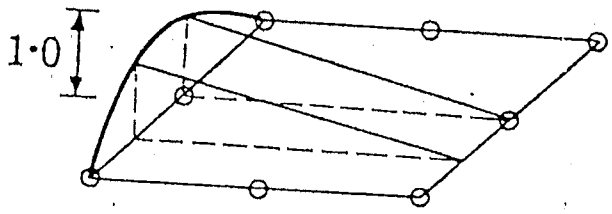


$N_5 = l_2^3(\xi)\, l_1^1(\eta)$ $N_6 = l_3^3(\xi)\, l_1^1(\eta)$ $N_7 = l_1^1(\xi)\, l_2^2(\eta)$

$N_1 = \hat{N}_1 - \frac{2}{3}N_5 - \frac{1}{3}N_6 - \frac{1}{2}N_5$, $N_2 = \hat{N}_2 - \frac{1}{3}N_5 - \frac{2}{3}N_6$

$N_3 = \hat{N}_3$, $N_4 = \hat{N}_4 - \frac{1}{2}N_7$

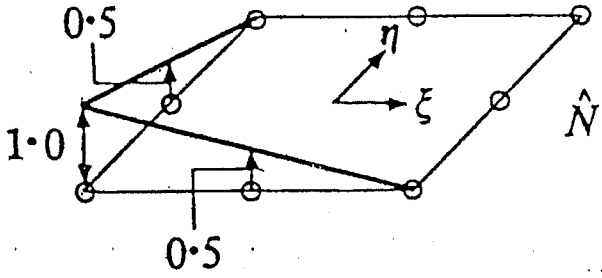




(*a*) $N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$ $\quad \ell_2^2(\xi)\,\bar{\ell}_1^1(\eta)$

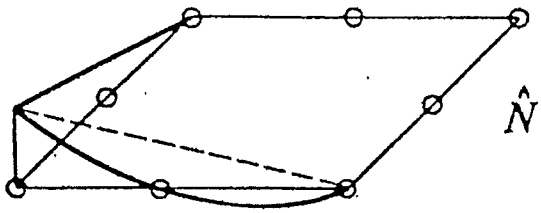
(*b*) $N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$ $\quad \ell_1^1(\xi)\,\ell_2^2(\eta)$

Step 1



$\hat{N}_1 = (1-\xi)(1-\eta)/4$

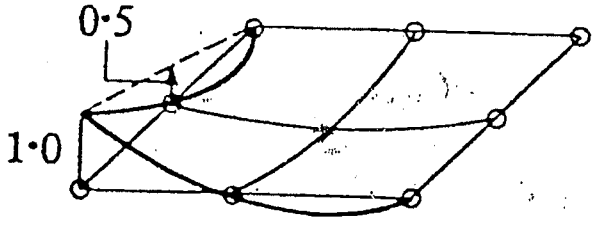
Step 2



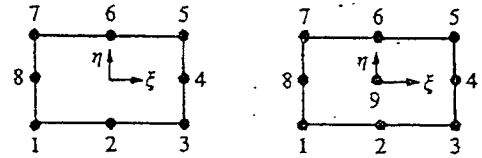
$\hat{N}_1 - \frac{1}{2} N_5$

(*c*)

Step 3



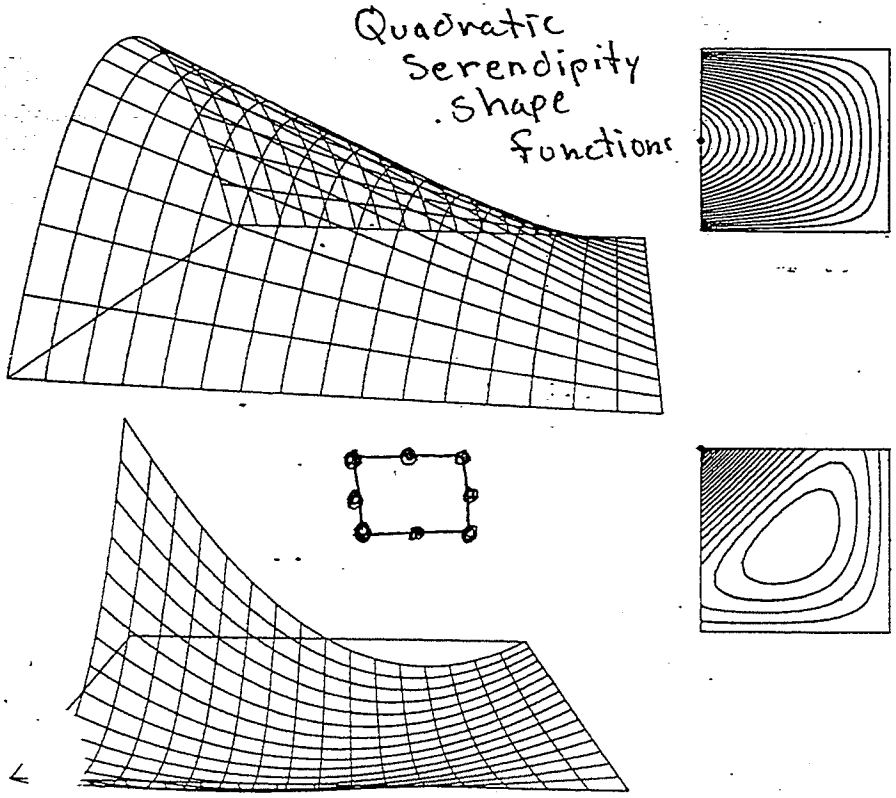
$N_1 = \hat{N}_1 - \frac{1}{2} N_5 - \frac{1}{2} N_8$



8-node Serendipity element 9-node Lagrangian element

Local node number	ξ_i	η_i
1	−1	−1
2	0	−1
3	1	−1
4	1	0
5	1	1
6	0	1
7	−1	1
8	−1	0
9	0	0

Fig. 8.7 Quadratic 8 and 9–node rectangular elements.



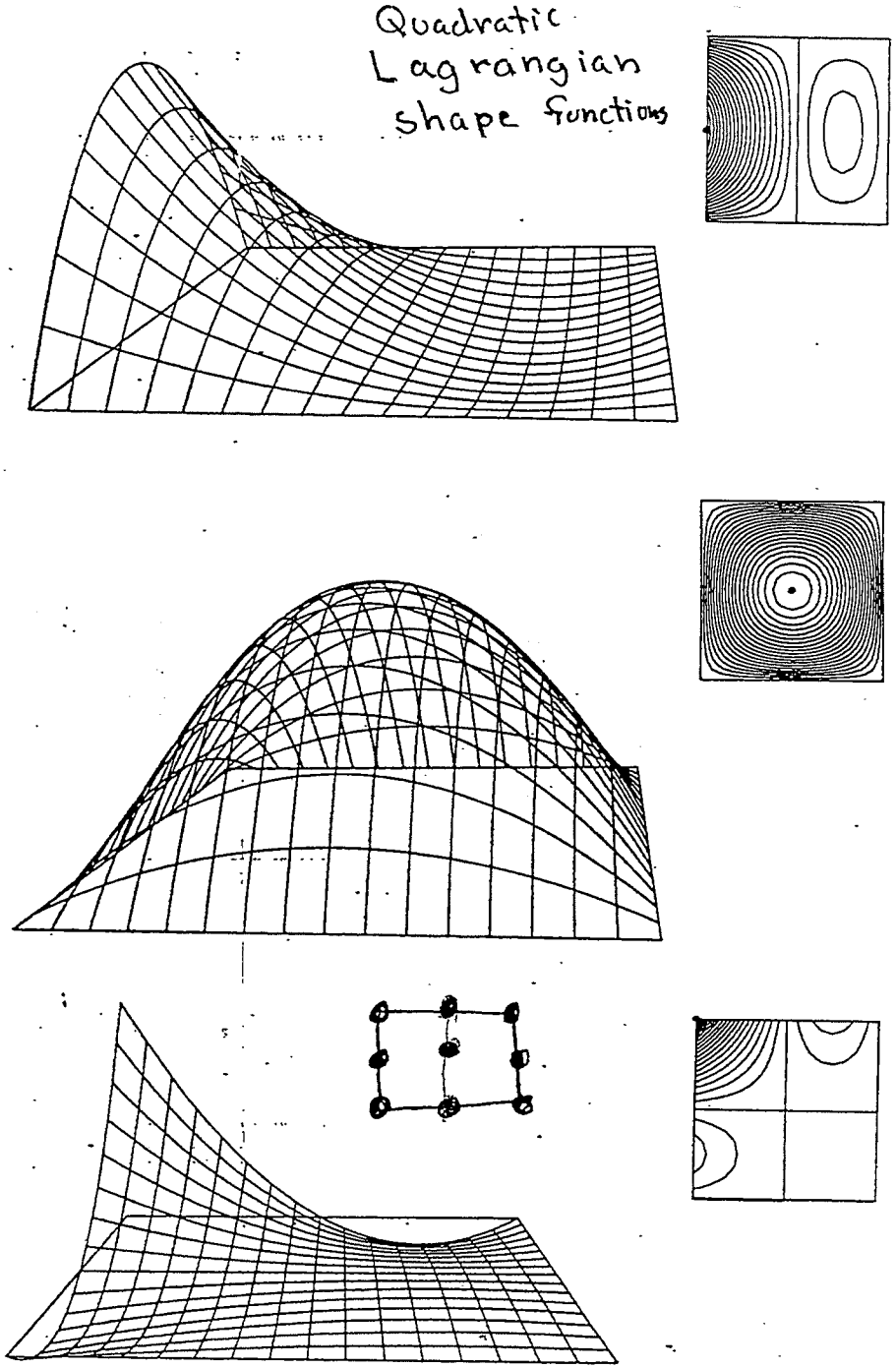
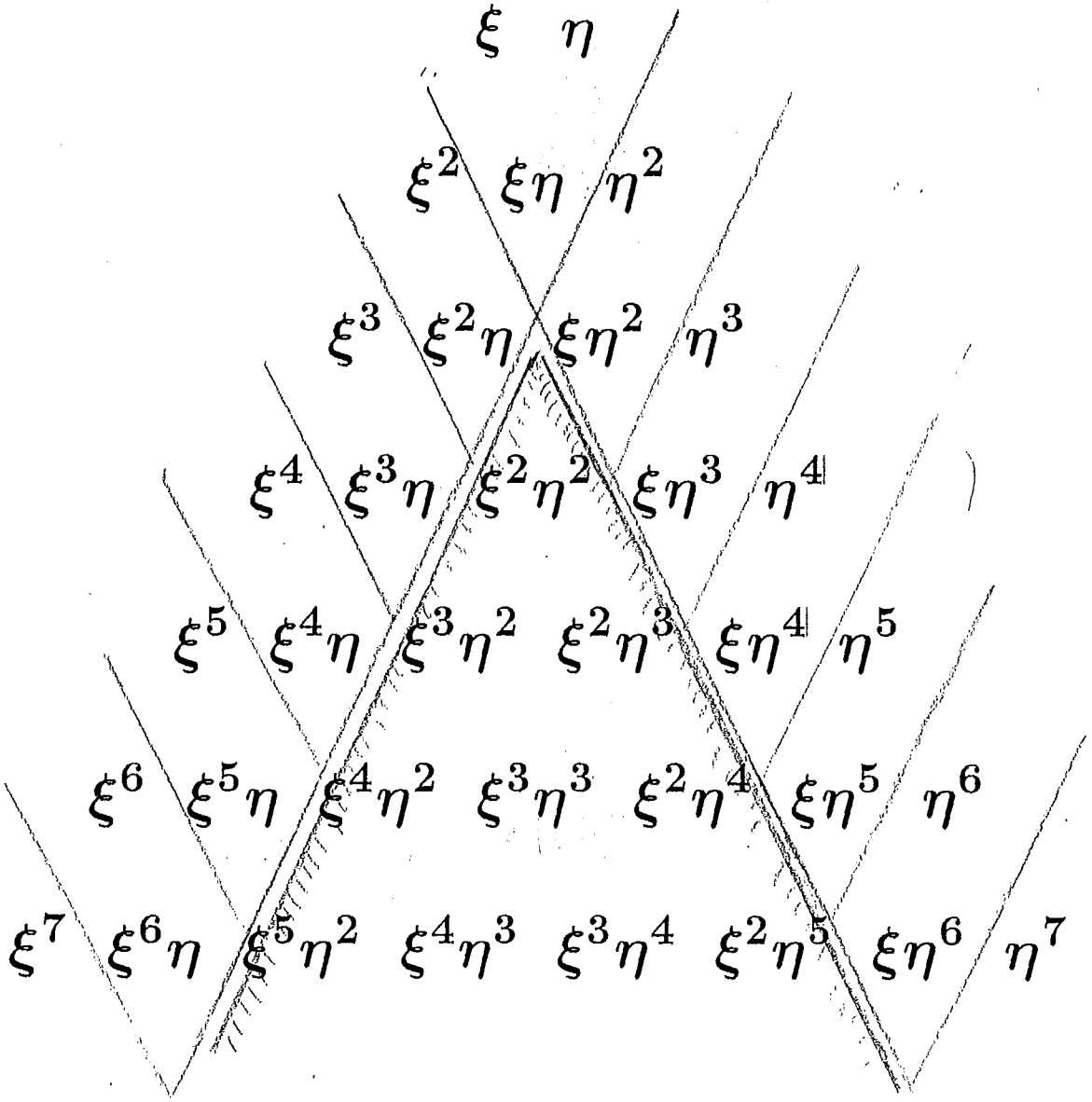


Fig. 8.10 Typical shape functions for the 9-node rectangular element.

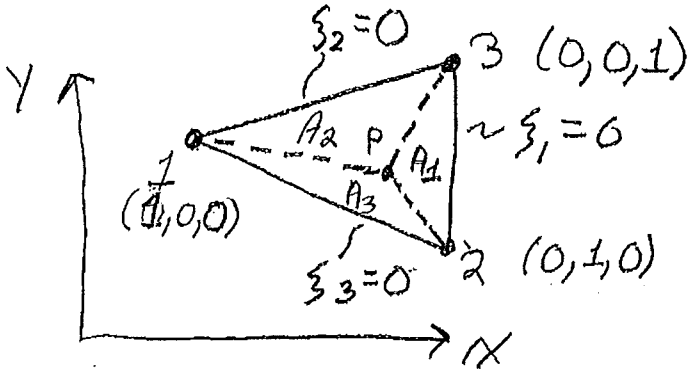
IM-23

Serendipity Elements

$$1$$



Area Coordinates and Triangular elements



$$\xi_i = \frac{A_i}{A}$$

A - Area of triangle

A_i - Area of subtriangle i

$$\xi_1 + \xi_2 + \xi_3 = \frac{A_1 + A_2 + A_3}{A} = \frac{A}{A} = 1$$

Only two independent coordinates -
At some point one need to eliminate on (ex. $\xi_3 = 1 - \xi_1 - \xi_2$)

Note - The area coordinates on a straight sided element define a convenient linear interpolant

$$u^h = \sum_{a=1}^{3} \xi_a d_a \qquad d_a = u(\underset{\sim}{x}_a)$$

For an isoparametric mapping

$$\underset{\sim}{x} = \sum_{a=1}^{3} \xi_a \underset{\sim}{x}_a \;, \quad x = \sum_{a=1}^{3} \xi_a x_a \quad y = \sum_{a=1}^{3} \xi_a y_a$$

In the case of straight sided triangles the mapping between $\underset{\sim}{x}$ and $\underset{\sim}{\xi}$ is linear and easily invertable.

This will not be the case for curved elements

In the general curved case we again note that we will use a map of $\underset{\sim}{x}$ in terms of $\underset{\sim}{\xi}$

recalling the only two of the area coordinates are independent the following is written assuming you selected ξ_1 and ξ_2 as the independent coordinates and every where there is a ξ_3 you replace it with

$$\xi_3 = 1 - \xi_1 - \xi_2$$

With this we apply chain rule knowing we can take $\partial/\partial\xi_1$ and $\partial/\partial\xi_2$ of things and invert what we have to produce

$$\partial/\partial x = \frac{1}{|J|}\left(\frac{\partial y}{\partial \xi_2}\frac{\partial}{\partial \xi_1} - \frac{\partial y}{\partial \xi_1}\frac{\partial}{\partial \xi_2}\right)$$

$$\partial/\partial y = \frac{1}{|J|}\left(-\frac{\partial x}{\partial \xi_2}\frac{\partial}{\partial \xi_1} + \frac{\partial x}{\partial \xi_1}\frac{\partial}{\partial \xi_2}\right)$$

$$|J| = \frac{\partial x}{\partial \xi_1}\frac{\partial y}{\partial \xi_2} - \frac{\partial y}{\partial \xi_1}\frac{\partial x}{\partial \xi_2}$$

We also have integration of $\int_{\Omega^e} f(\xi_1,\xi_2,\xi_3)\,d\Omega$ again using $\xi_3 = 1-\xi_1-\xi_2$ we have

$$\int_{\Omega^e} f(\xi_1,\xi_2)\,d\Omega = \int_{\hat{\Omega}^e} f(\xi_1,\xi_2)\,|J|\,d\xi_1\,d\xi_2$$

$\hat{\Omega}^e$ parametric triangle

In the case of straight sided elements the expressions are easier

$$|J| = 2A$$

and integrals of $\xi_1^\alpha \xi_2^\beta$ have closed form expressions

$$\int_{\Omega^e} \xi_1^\alpha \xi_2^\beta \, d\Omega = 2A \, \frac{\alpha!\,\beta!}{(\alpha+\beta+2)!}$$

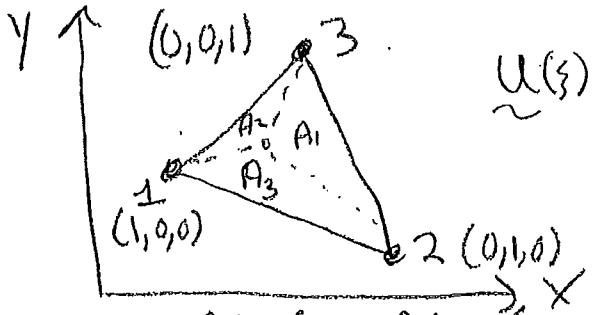
The selection of polynomial terms for triangles will naturally follow the Pascal's triangle -

The interpolating triangles are -

3- nodes → linear
6- nodes → quadratic
10 - nodes → cubic
etc.

3-noded linear, isoparametric triangle

⇒ Directly interpolate using the area coordinates $N_a = \xi_a = \frac{A_a}{A}$ $a = 1(2)3$



$$\underset{\sim}{u}(\xi) = \sum_{a=1}^{3} N_a(\xi)\,\underset{\sim}{u}_a$$

$$\underset{\sim}{x}(\xi) = \sum_{a=1}^{3} N_a(\xi)\,\underset{\sim}{x}_a$$

$N_1 = \xi_1$, $N_2 = \xi_2$, $N_3 = \xi_3 = 1 - \xi_1 - \xi_2$

T-5

$$1$$

$$\xi \quad \eta$$

$$\xi^2 \quad \xi\eta \quad \eta^2$$

$$\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$$

$$\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4$$

$$\xi^5 \quad \xi^4\eta \quad \xi^3\eta^2 \quad \xi^2\eta^3 \quad \xi\eta^4 \quad \eta^5$$

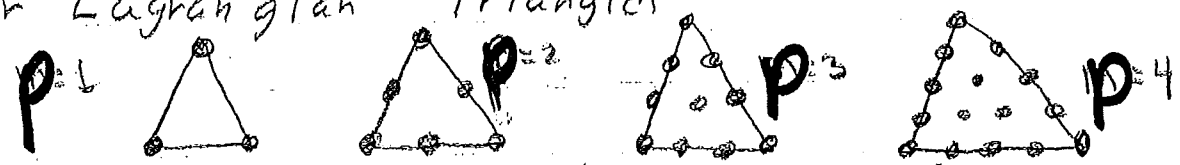
$$\xi^6 \quad \xi^5\eta \quad \xi^4\eta^2 \quad \xi^3\eta^3 \quad \xi^2\eta^4 \quad \xi\eta^5 \quad \eta^6$$

$$\xi^7 \quad \xi^6\eta \quad \xi^5\eta^2 \quad \xi^4\eta^3 \quad \xi^3\eta^4 \quad \xi^2\eta^5 \quad \xi\eta^6 \quad \eta^7$$

Interpolation for triangles

Note: Expressions differ in notation from that in Appendix 3.I, result the same

For Lagrangian triangles



Linear quadratic Cubic Quartic

complete polynomials of order p

have $n = p+1$ nodes on each side

$$u = \sum_{a=1}^{n_{en}} N_a d_a \qquad n_{en} = \frac{1}{2}(p+1)(p+2)$$

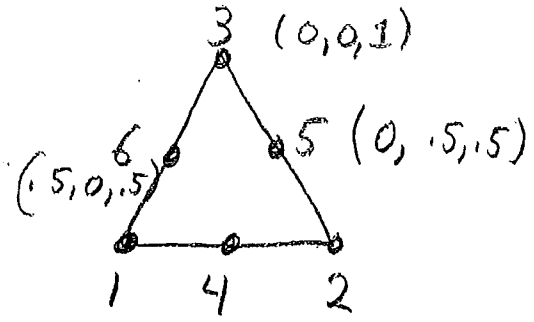
# of terms in complete p^{th} order polynomial

$$N_a(\xi) = \prod_{i=1}^{3} N_i^a(\xi_i) = N_1^a(\xi_1) N_2^a(\xi_2) N_3^a(\xi_3)$$

$$N_i^a(\xi_i) = \begin{cases} \prod_{J=1}^{I_i^a} \left(\dfrac{p\xi_i - J + 1}{J} \right) & \text{for } I_i^a \geq 1 \\ 1 & \text{for } I_i^a = 0 \end{cases}$$

$$I_i^a = p\,\xi_i \Big|_{\xi = \xi_a}$$ ⇐ Second term is the ξ_i coordinate of node a

$p=2$



$N_1 = N_1'(\xi_1) N_2'(\xi_2) N_3'(\xi_3)$

$I_1^1 = 2(1) = 2$, $I_2^1 = 2(0) = 0$, $I_3^1 = 2(0) = 0$

$N_2^1 = 1$, $N_3^1 = 1$, $N_1^1 = \prod_{J=1}^{2} \left(\frac{2\xi_1 - J + 1}{J}\right) = 2\xi_1 \left(\frac{2\xi_1 - 1}{2}\right) = \xi_1(2\xi_1 - 1)$

$N_1 = \xi_1(2\xi_1 - 1)(1)(1) = \xi_1(2\xi_1 - 1)$

$N_2 = N_1^2(\xi_1) N_2^2(\xi_2) N_3^2(\xi_3)$, $I_1^2 = 0 = I_3^2$, $I_2^2 = 2$

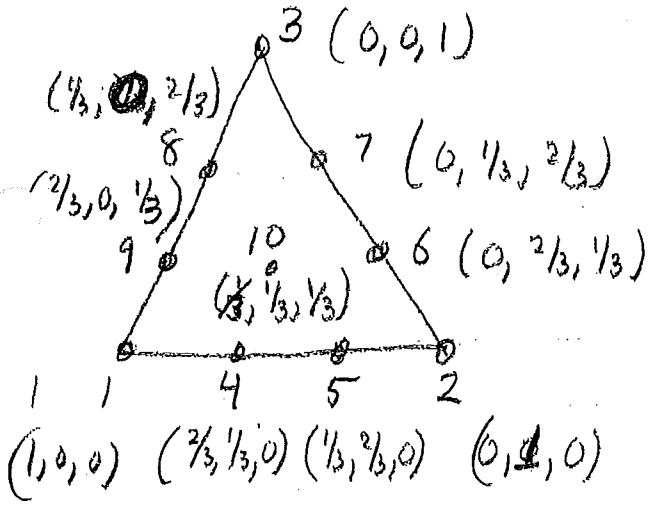
$N_1^2 = 1$, $N_3^2 = 1$, $N_2^2 = \prod_{J=1}^{2} \left(\frac{2\xi_2 - J + 1}{J}\right) = \xi_2(2\xi_2 - 1)$

$N_2 = (1)\,\xi_2(2\xi_2 - 1)\,(1) = \xrightarrow{J=1} \xi_2(2\xi_2 - 1)$

$N_4 = N_1^4(\xi_1) N_2^4(\xi_2) N_3^4(\xi_3)$, $I_1^4 = 2(.5) = 1$, $I_2^4 = 2(.5) = 1$, $I_3^4 = 0$

$N_1^4 = \prod_{J=1}^{1} \left(\frac{2\xi_1 - J + 1}{J}\right) = 2\xi_1$, $N_2^4 = 2\xi_2$, $N_3^4 = 1$

$N_4 = 4\xi_1\xi_2$



$p = 3$

$N_1 = N_1^1(\xi_1)\, N_2^1(\xi_2)\, N_3^1(\xi_3)$, $I_1^1 = 3(1) = 3$, $I_2^1 = 0$, $I_3^1 = 0$

$$N_J^1 = \prod_{J=1}^{3} \left(\frac{3\xi_1 - J + 1}{J} \right) = 3\xi_1 \left(\frac{3\xi_1 - 1}{2} \right) \left(\frac{3\xi_1 - 2}{3} \right) = \frac{\xi_1}{2} (3\xi_1 - 1)(3\xi_1 - 2)$$

$$N_1 = N_1^1 \cancel{N_2^1 N_3^1} = \frac{\xi_1}{2} (3\xi_1 - 1)(3\xi_1 - 2)$$

$N_4 = N_1^4(\xi_1)\, N_2^4(\xi_2)\, N_3^4(\xi_3)$, $I_1^4 = 3(2/3) = 2$, $I_2^4 = 3(1/3) = 1$, $I_3^4 = 0$

$$N_1^4 = \prod_{J=1}^{2} \left(\frac{3\xi_1 - J + 1}{J} \right) = 3\xi_1 \left(\frac{3\xi_1 - 1}{2} \right), \quad N_3^4 = 1$$

$$N_2^{4,1} = \prod_{J=1}^{1} \left(\frac{3\xi_2 - J + 1}{J} \right) = 3\xi_2$$

$$N_4 = 3\xi_1 \left(\frac{3\xi_1 - 1}{2} \right) 3\xi_2 = \frac{9}{2} \xi_1 \xi_2 (3\xi_1 - 1)$$

$N_{10} = N_1^{10}(\xi_1)\, N_2^{10}(\xi_2)\, N_3^{10}(\xi_3)$, $I_i^{10} = 3(1/3) = 1$

$$N_i^{10} = \prod_{J=1}^{1} \left(\frac{3\xi_i - J + 1}{J} \right) = 3\xi_i$$

$$N_{10} = 3\xi_1\, 3\xi_2 3\xi_3 = 27 \xi_1 \xi_2 \xi_3$$

Numerical Integration

References:

- Numerical Analysis, Burden and Faires, Pws-Kent Publishing Co.
- F.E. Procedures in Engineering Analysis, K.J. Bathe, Prentice Hall

We have seen that in many cases the integrand to our element stiffness integral is a rational polynomial. Thus we are not likely to have exact integration formula for the integrals. Thus we must integrate numerically in that case. Even if when we can integrate exactly it is typically more convenient and just as efficient, to use numerical integration.

Numerical quadrature - Look at 1-D

$$\int_{-1}^{1} g(\xi)\, d\xi = \sum_{\ell=1}^{n_{int}} w_\ell\, g(\xi_\ell) + R_{n_{int}-1}$$

How do we get to such a formula - will use an interpolating polynomial to approximate the integrand and integrate those terms to obtain our weights. Using our Lagrange polynomials

$$g(\xi) = \sum_{\ell=1}^{n_{int}} g(\xi_\ell)\, L_\ell^{n_{int}-1}(\xi) + \tilde{R}$$

in which case our integral becomes

$$\int_{-1}^{1} g(\xi)\,d\xi = \int_{-1}^{1} \sum_{\ell=1}^{n_{int}} g(\xi_\ell)\, L_\ell^{n_{int}-1}\, d\xi + \int_{-1}^{1} \hat{R}\, d\xi$$

$$= \sum_{\ell=1}^{n_{int}} \left(g(\xi_\ell) \int_{-1}^{1} L_\ell^{n_{int}-1}\, d\xi \right) + R_{n_{int}-1} = \sum_{\ell=1}^{n_{int}} w_\ell\, g(\xi_\ell) + R_{n_{int}-1}$$

$\Rightarrow$ note $-\ w_\ell = \int_{-1}^{1} L_\ell^{n_{int}-1}\, d\xi$

It can be shown that at some location $-1 \le \hat{\xi} \le 1$

$$\hat{R} = \left(\frac{d^{n_{int}} g(\hat{\xi})}{d\xi^{n_{int}}} \Big/ n_{int}! \right) \prod_{\ell=1}^{n_{int}} (\xi - \xi_\ell)$$

$$R_{n_{int}-1} = \int_{-1}^{1} \hat{R}\, d\xi$$

This method is referred to as Newton-Cotes

Two common Newton Cotes rules:

Trapezoidal Rule, $n_{int} = 2$, points at $\xi = -1, \xi = 1$

$$w_1 = \int_{-1}^{1} l_1^1\, d\xi = \int_{-1}^{1} \tfrac{1}{2}(1-\xi)\, d\xi = 1$$

$$w_2 = \int_{-1}^{1} l_2^1\, d\xi = \int_{-1}^{1} \tfrac{1}{2}(1+\xi)\, d\xi = 1$$

$$R = -\tfrac{2}{3}\, g_{,\xi\xi}(\hat{\xi}) \Leftarrow$$ We will exactly integrate a linear

Simpson's Rule, $n_{int}=3$, points at $\xi=-1, \xi=0, \xi=1$

$$w_1 = \int_{-1}^{1} l_1^2 d\xi = \int_{-1}^{1} \frac{1}{2}(\xi^2 - \xi)\, d\xi = 1/3$$

$$w_2 = \int_{-1}^{1} l_2^2 d\xi = 4/3 \quad , \quad w_3 = \int_{-1}^{1} l_3^2 d\xi = 1/3$$

The error equation we saw before would say the error is $\alpha\, g_{,\xi\xi\xi}$

However, a more careful analysis shows that the error is:

$$R = \frac{-8 f''''(\xi)}{90}$$ ← proportional to 4th derivative

This says cubics would be integrated exactly

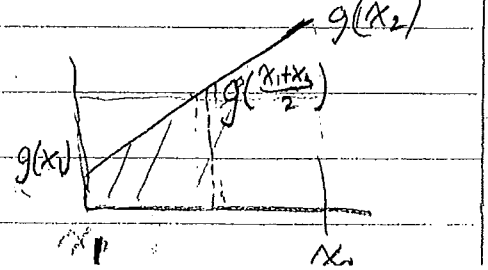
How can that be - We only used 3 conditions (values at 3 points) - So why do we do better than 3-1=2 for the order curve done exactly?

⇒ The positions of the evaluation points in the numerical integration can be used as conditions - In this case one point (the center point) is at such an "optimal" location

There is a procedure to find these points which leads to the Guass Quadrature ~~Rules~~ where with n_{int} integration points you can exactly integrate an order $2n_{int}-1$ order polynomial

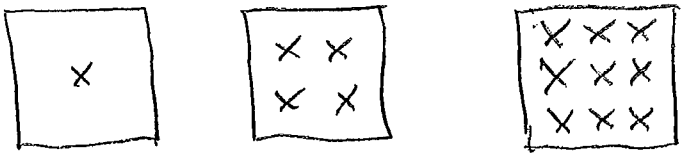
⇒ Note - we are not saying we exactly represent it - just exactly integrate it.

Consider a linear curve - Can easily exactly integrate using average value (center value) times interval → $A = g\left(\frac{x_1+x_2}{2}\right)(x_2-x_1)$



Integration for $n_{sd} > 1$

For Quads and Hex elements
Use products of 1-D rules



$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\eta)\,d\xi\,d\eta \approx \int_{-1}^{1} \sum_{i=1}^{n_{int_\xi}} g(\xi_i,\eta)\,w_i\,d\eta$$

$$= \sum_{j=1}^{n_{int_\eta}} \sum_{i=1}^{n_{int_\xi}} g(\xi_i,\eta_j)\,w_i w_j = \sum_{k=1}^{n_{int}} w_k\, g(\xi_k,\eta_k)$$

$$n_{int} = n_{int_\xi}\, n_{int_\eta}$$

For triangles and Tets
A couple of options -

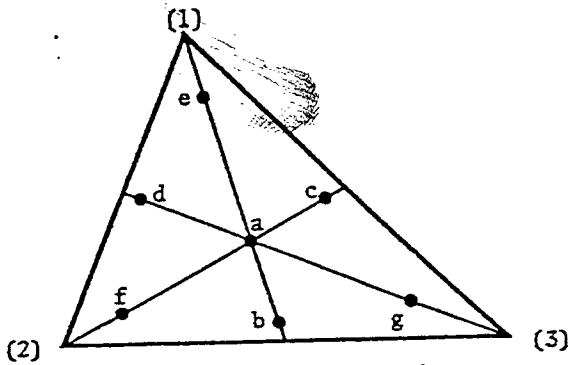
(A) Degenerate product rule

(B) Specifically constructed single loop Simplex integration rules

Hammer's Rule Common for triangles

$$\int_{\Delta} g(\xi_1,\xi_2,\xi_3)\,d\Delta = \sum_{k=1}^{n_{int}} g(\xi_{1_k},\xi_{2_k},\xi_{3_k})\,w_k$$

SEVEN POINT FIFTH ORDER INTEGRATION



	ξ_1	ξ_2	ξ_3	w
a	1/3	1/3	1/3	9/80
b	α_1	β_1	β_1	η_1
c	β_1	α_1	β_1	η_1
d	β_1	β_1	α_1	η_1
e	α_2	β_2	β_2	η_2
f	β_2	α_2	β_2	η_2
g	β_2	β_2	α_2	η_2

$\alpha_1 = 0.05971\ 58717\ 89770$

$\beta_1 = 0.47014\ 20641\ 05115$

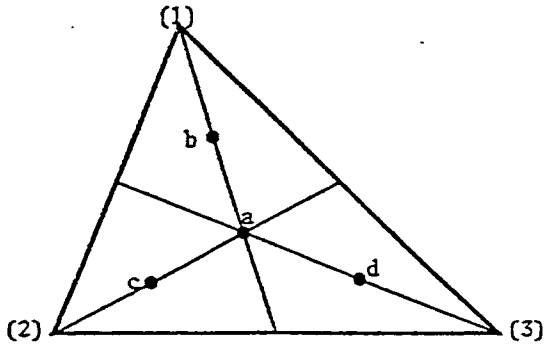
$\eta_1 = 0.06619\ 70763\ 94253$

$\alpha_2 = 0.79742\ 69853\ 53087$

$\beta_2 = 0.10128\ 65073\ 23456$

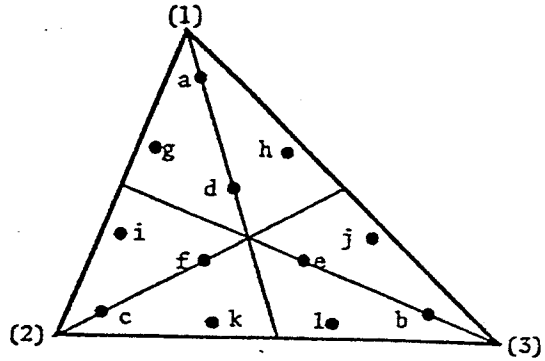
$\eta_2 = 0.06296\ 95902\ 72414$

FOUR POINT CUBIC INTEGRATION



	ξ_1	ξ_2	ξ_3	w
a	1/3	1/3	1/3	-27/96
b	3/5	1/5	1/5	25/96
c	1/5	3/5	1/5	25/96
d	1/5	1/5	3/5	25/96

TWELVE POINT SIXTH ORDER INTEGRATION



	ξ_1	ξ_2	ξ_3	w
a	α_1	β_1	β_1	η_1
b	β_1	β_1	α_1	η_1
c	β_1	α_1	β_1	η_1
d	α_2	β_2	β_2	η_2
e	β_2	β_2	α_2	η_2
f	β_2	α_2	β_2	η_2
g	α_3	β_3	γ_3	η_3
h	α_3	γ_3	β_3	η_3
i	β_3	α_3	γ_3	η_3
j	β_3	γ_3	α_3	η_3
k	γ_3	α_3	β_3	η_3
l	γ_3	β_3	α_3	η_3

$\alpha_1 = 0.87382\ 19710\ 16996$

$\beta_1 = 0.06308\ 90144\ 91502$

$\eta_1 = 0.02542\ 24531\ 85104$

$\alpha_2 = 0.50142\ 65096\ 58179$

$\beta_2 = 0.24928\ 67451\ 70910$

$\eta_2 = 0.05839\ 31378\ 63190$

$\alpha_3 = 0.63650\ 24991\ 21399$

$\beta_3 = 0.31035\ 24510\ 33785$

$\gamma_3 = 0.05314\ 50498\ 44816$

$\eta_3 = 0.04142\ 55378\ 09187$

What order integration should we use?

Remember for generally shaped elements the integrand is a rational polynomial thus our numerical integration can not be exact. Even when the $|J|$ is constant, which means we would have polynomials (with polynomial shape functions) we may not want to integrate exactly.

So if you do not integrate exactly- How well do you want to integrate?

Take this in two levels

First level - Integrate well enough that our F.E. method will execute and, in the limit, Converge to the exact solution.

Second level - To integrate well enough that we converge at the maximum possible rate as dictated by the highest order complete polynomial in our shape functions.

First Level - Integrate well enough

Two considerations
- to maintain convergence (minimal rate)
- to avoid a singular stiffness matrix

Maintain Convergence (minimal rate)

(min.) Convergence can be maintained so long as we can integrate a constant value of the m^{th} derivative as it appears in the energy inner product correctly

That is we want to correctly integrate

$$\int_{\Omega} C \, d\Omega = \int_{\square} C|J| \, d\square = C \int_{\square} |J| \, d\square$$

Thus we have to integrate the element volume correctly. For polynomial shape functions and isoparametric mappings $|J|$ is a polynomial-

Thus we need to integrate this polynomial exactly to get the correct Volume -

This defines a minimum order of integration possible.

To Avoid a Singular Stiffness Matrix

we have to solve $\underset{\sim}{K}\underset{\sim}{d}=\underset{\sim}{F}$

if the stiffness matrix is not full rank then $\underset{\sim}{K}^{-1}$ does not exist and we can not solve the system

If we use too low an order of numerical integration we will not have sufficient conditions and we would find our elements have too many zero eigen values with one or more corresponding to a valid (energy producing if properly integrated eigenmode)

To look at this lets consider 2-D elasticity and individual element matrices

$$\underset{\sim}{K}^{e}_{n_{ed}\times n_{ed}}$$

Note since the element has no essential boundary conditions an exactly integrated element stiffness matrix will have 3 zero eigen values, the eigen vectors of which correspond to the 3 rigid body modes

To Converge at maximum rate possible

For problems with smooth solutions (no singularities)

$$\|u^h - u\|_m \leq \bar{C} h^{k+1-m} \|u\|_{k+1}$$

m - highest order derivative in $a(w,u)$
k - highest order complete polynomial in u^h

To maintain the convergence rate of $k+1-m$ must use a high enough order integration rule to exactly integrate a polynomial of order

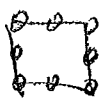
$$(\bar{k} + k - 2m)$$

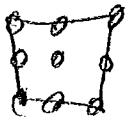
$\bar{k}$ - highest order monomial term appearing in u^h

 - $\bar{k}=1$

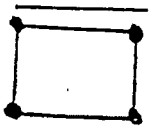
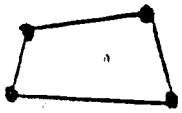
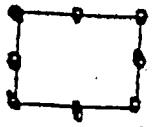
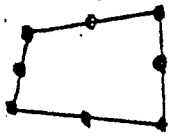
 - $\bar{k}=2$ ($\xi\eta$ term)

 - $\bar{k}=2$

 - $\bar{k}=3$ ($\xi^2\eta$ and $\xi\eta^2$ terms)

 - $\bar{k}=4$ ($\xi^2\eta^2$ term)

ORDERS OF GAUSSIAN INTEGRATION FOR TWO DIMENSIONAL ELEMENTS

ELEMENT	DISPLACEMENT MODEL	GEOMETRY MODEL	MINIMUM ORDER [a,b,c]	USUAL OR RECOMMENDED ORDER [a,d]	EXACT OR MAXIMUM ORDER [e]
	LINEAR	RECTANGULAR	1x1	2x2	2x2
	LINEAR	LINEAR	1x1	2x2	3x3
	QUADRATIC	RECTANGULAR	2x2	2x2	3x3
	QUADRATIC	LINEAR	2x2	2x2	3x3
	QUADRATIC	QUADRATIC	3x3	3x3	4x4

NOTES: (A) REFERENCE: ZIENKIEWICZ & HINTON, "REDUCED INTEGRATION, FUNCTION SMOOTHING AND NON-CONFORMITY IN FINITE ELEMENT ANALYSIS", PROC. 1976 INT. CONF. ON FINITE ELEMENT METHODS IN ENGINEERING, UNIV. OF ADELAIDE, AUSTRALIA, AND ALSO TO BE PUBLISHED IN THE JOURNAL OF THE FRANKLIN INSTITUTE.

(B) ZERO ENERGY DEFORMATION MODES MAY ARISE. IF THESE ARE NOT INTERELEMENT COMPATIBLE, THEN THE RECOMMENDED ORDER IS HIGHER THAN THE MINIMUM ORDER.

(C) BASED ON VOLUME-INTEGRATION CRITERION: COOK, SECTION 5.7

(D) REFERENCE: BATHE & WILSON, SECTION 4.7.

(E) ALL VALUES GIVEN ARE FOR CONSTANT THICKNESS ELEMENTS.