BRIEF REVIEW OF NUMERICAL SOLUTIONS TO PROBLEMS IN MATHEMATICAL PHYSICS BY THE FINITE ELEMENT METHOD

Starting Point - A statement of the basic properties of a physical system to which information is needed about how that system will react to a given set of conditions. Some standard examples:

1. Given an object made of a specific material, supported in a given manner and subjected to specific sets of mechanical loads, what is the peak displacement, the peak stress, the likelihood of fracture in a given time period, etc.?
2. What are the variations and distributions associated with the above parameters given the variations and distributions of the material properties and loads?
3. Given a known distribution of soils, ground water and hydrocarbon pollutants, determine the number and location of wells for the injection of bacteria and air to most effectively have the hydrocarbon pollutants decomposed in a fixed period of time.
4. What is the best shape and material to use for a dental implant that will maintain its bond to the bone?

Basic steps that must be performed to solve the problem.

1. Determine an appropriate mathematical model that governs the physical problem. This process typically includes a number of specific idealizations to make the problem tractable.

   Consider the statement of a problem in mathematical physics on a given domain. The strong statement of a problem is in the form of a set of governing partial differential equations as:

   Given $\boldsymbol{f} : \Omega \times]0,T[\rightarrow R$ and needed conditions, $\mathbf{u}_0$, $\mathbf{g}$, $\mathbf{h}$, and coefficients in D( ) and β( );

   find $\mathbf{u} : \Omega \times]0,T[\rightarrow R$ such that

$$D(\mathbf{u}) = \mathbf{f} \quad \text{on} \quad \Omega \times]0,T[\tag{1}$$

subject to

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0 \qquad \mathbf{x} \in \Omega \tag{2a}$$

$$\mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma_g \times]0,T[\tag{2b}$$

$$\beta(\mathbf{u}) = \mathbf{h} \quad \text{on} \quad \Gamma_h \times]0,T[\tag{2c}$$

$$\Gamma = \Gamma_g \cup \Gamma_h$$

where

D(**u**)	is a differential operator acting on the dependent variable **u**.
u	is the dependent variable which may have several components. It is the solution to **u** that is being sought.
f	is the 'forcing function' driving the problem.
x	independent spatial variables.
t	independent temporal variable.
0, T	initial and final times.
Ω	the domain of interest.
u(**x**,0)	initial prescribed values of the dependent variables at the starting time.

g prescribed essential boundary conditions.

Γ_g portion of the domain boundary with prescribed essential boundary conditions.

$\beta(\mathbf{u})$ differential operator that yield the natural boundary terms when evaluated on the boundary.

h prescribed natural boundary conditions.

Γ_h portion of the domain boundary with prescribed natural boundary conditions.

Γ the boundary of the domain Ω.

The statement of the boundary conditions and initial conditions must be consistent with the differential operator in the domain, D( ), having second partial derivatives of the spatial coordinates and first partial derivatives with respect to time. In this case, the differential operator in the natural boundary condition equation, $\beta(\ )$, has first derivatives in the spatial coordinates. Domain operators with higher order derivatives can be handled in the same basic manner, except that the boundary condition terms and the initial conditions would deal with higher derivative terms.

2. Develop a problem definition - this includes the domain of the analysis and the analysis attribute information. In general there are typically a number of idealization approximations that are involved with this step. Historically the approximation errors associated with them are considered part of engineering judgement and not controlled explicitly. Research efforts are beginning to address these issues. We may discuss this briefly some time this semester.

   The domain definition is not the finite element mesh. It is the geometric description of the analysis domain. It is the domain that will be passed to the mesh generator to develop the finite element discretization. Typically this will be housed in electronic form in the data structures of a geometric modeling system.

   Analysis attribute data is that information, beyond the geometric domain definition, that is needed to qualify the physical problem to be solved. Analysis attribute information includes material properties, boundary conditions, loads and initial conditions. To see the flexibility of a general statement, consider the physical problems defined by Poisson's equation and associated boundary conditions which can be written as:

$$\nabla(\boldsymbol{\kappa}\nabla\mathbf{u}) - \boldsymbol{f} = 0 \quad \mathbf{x} \in \Omega \tag{3}$$

subject to

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_g \tag{4a}$$

$$\boldsymbol{\kappa}\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \lambda\mathbf{u} = \mathbf{h} \quad \text{on } \Gamma_h \tag{4b}$$

$$\Gamma = \Gamma_g \cup \Gamma_h$$

where $\boldsymbol{\kappa}$ and λ are material parameters and $\mathbf{n}$ is the normal direction. Problem classes governed by this equation include:

a. Heat transfer where $\mathbf{u}$ is the temperature, $\boldsymbol{\kappa}$ is the conductivity, $\boldsymbol{f}$ is the heat source, λ is the convection constant, and $\mathbf{h}$ is heat flow.

b. Irrotational flow of an ideal fluid where **u** is a stream function, **κ** is density, ***f*** is mass production, and first partials of **u** are flow components.
c. Ground-water flow where **u** is the piezometric head, **κ** is permeability, ***f*** is recharge, and **h** is seepage.
d. Torsion on a cross-section where **u** is a stress function, **κ** is the torsional flexibility, ***f*** is twist, and first partials of **u** are stress components.
e. Electrostatics where **u** is a scalar potential, **κ** is the dielectric constant, and ***f*** is the charge density.
f. Magnetostatics where **u** is a magnetic potential, **κ** is the magnetic permeability, and ***f*** is the charge density.
g. Transverse deflection of an elastic membrane where **u** is the transverse deflection, **κ** is the tension in the membrane, ***f*** is the transversely distributed load and **h** is the normal force.

3. Carry out domain simplification and dimensional reduction idealizations. Things here include ignoring small holes and fillets, using beams and shells, etc. Dimensional reductions can lead to altered forms of the governing equations.

4. Develop a weak form for the problem. For example use a weighted residual of the form

$$\int_{\Omega\times]0,T[} \mathbf{w}(D(\mathbf{u}) - \mathbf{f})\,d\Omega = 0 \quad \text{on } \Omega \times]0,T[\tag{5}$$

where
$\mathbf{u} \in \delta$ and $\mathbf{w} \in V$, with δ and V being the appropriate spaces.
Continue the process by carrying out appropriate integrations by parts to obtain the needed boundary condition information back and get the a priori conditions on **u** and **w** such that the spaces δ and V are of the desired form. Symbolically this form is:

$$a(\mathbf{w},\mathbf{u}) = (\mathbf{w},\boldsymbol{f}) + (\mathbf{w},\mathbf{h})_\Gamma \tag{6}$$

5. Select the appropriate finite dimensional spaces and rewrite the weak form with account taken to the essential boundary conditions such that the piecewise polynomial functions written on an elemental level can meet the needed a priori conditions.

$$\mathbf{u}^h = \mathbf{v}^h + \mathbf{g}^h \tag{7}$$

which yields

$$a(\mathbf{w}^h,\mathbf{v}^h) = (\mathbf{w}^h,\boldsymbol{f}) + (\mathbf{w}^h,\mathbf{h})_\Gamma - a(\mathbf{w}^h,\mathbf{g}^h)$$

6. Approximate the functions $\mathbf{v}^h$, and $\mathbf{w}^h$, and interpolate $\mathbf{g}^h$ in terms of piecewise polynomial functions written on an elemental level that can meet the needed a priori conditions. For vector quantities this is preceded by a step to write the vectors in terms of components.

$$\mathbf{v}^h = v_i^h\ \mathbf{e}_i \tag{8a}$$

$$\mathbf{g}^h = g_i^h\ \mathbf{e}_i \tag{8b}$$

$$\mathbf{w}^h = w_i^h \ \mathbf{e}_i \tag{8c}$$

The components are then written in terms of shape functions as

$$v_i^h = \sum N_a d_{ia} \tag{9a}$$

$$g_i^h = \sum N_b g_{ib} \tag{9b}$$

$$w_i^h = \sum N_a c_{ia} \tag{9c}$$

7. Relate elemental dof to global dof, substitute into the weighted residual form to yield the element stiffness equation

$$\mathbf{k}^e \, d^e = f^e \tag{10}$$

8. Assemble element contributions to form the global stiffness matrix

$$\mathbf{K}d = f \tag{11}$$

9. Solve the algebraic equations

10. Recover secondary variables such as fluxes and stresses

11. Evaluate the discretization (and other idealization) errors and enrich the analysis model as needed. Repeat appropriate previous steps as needed. Exit this iteration process when solution accuracy to the prescribed degree of accuracy is obtained.

12. 'Postprocess' results to make as presentable and accurate as possible. There are techniques to postprocess secondary variables to improve their overall accuracy.

13. Present results.