Components, of an adaptive analysis
- Problem definition
- equation discretization and solution - the F.E. analysis
- error estimators / indicators
- Correction indicators
- Mesh improvement strategies
- Solve again and repeat the process

Error estimation

A wide array of references - most all papers with particular perspectives.

Mark Ainsworth and J. Tinsley Oden, A Posteriori Error Estimation in Finite Element Analysis, Wiley, 2000

Wolfgang Bangerth and Rolf Rannacher, Adaptive Finite Element Methods for Differential Equations, Birkhäuser, 2003

Rüdiger Verfürth, A Posteriori Error Estimation Techniques for Finite Element Methods, Oxford, 2013

Would like a tight bound the error in norms of interest. This is actually really hard, until you remember you are asking for the difference between the exact and F.E. solutions for a problem that you do not know the exact solution of.

A-priori techniques are an important starting point of the process, but they really only give the rate of convergence.

A-posteriori error estimators employ the finite element solution, u^h, in the process of getting an actual estimate

Consider our Finite Element Problem

Find $u^h \in S^h \subset S$ such that

$$a(w^h, u^h) = (w^h, f) + (w^h, h)_{\Gamma_h}, \quad \forall w^h \in V^h \subset V \qquad (1)$$

The error is $e = u - u^h$. Substitute $u = u^h + e$ in (1) and stating the infinite dimensional problem

$$a(w, e) = (w, f) + (w, h)_{\Gamma_h} - a(w, u^h) = R(w) \; ; \forall w \in V \qquad (2)$$

We can not solve (2) with out going to finite dimenisonal spaces-

$e \approx E \in S^*$ and $w^* \in V^*$ (both finite dimensional)

$$a(w^*, E) = (w^*, f) + (w^*, h)_{\Gamma_h} - a(w^*, u^h) \quad \forall w^* \in V^* \subset V$$

What happens is w^* and E are selected from S^h, w^h

$$\begin{aligned} a(w^*, E) &= (w^*, f) + (w^*, h)_{\Gamma_h} - a(w^*, u^h) \\ &= (w^h, f) + (w^h, h)_{\Gamma_h} - a(w^h, u^h) \\ &= \underline{\underline{0}} \end{aligned}$$

must use a richer space!

One can proceed with this and set-up and solve the stated problem - However, will be very expensive. Would like to define a more efficient calculation (will require some added approximation).

Would like to get a form that is a sum of element contributions -

- less expensive
- element level contributions can be used as part of correction indication
  - refine where high, coarsen where low

Lets apply divergence to eq (2) and separate into a set of element level problems

$$a(w,e)_{\Omega_i} = (w,f)_{\Omega_i} + (w,h)_{\Gamma_{hi}} - a(w,u^h)_{\Omega_i} + (w, \mathcal{J}(u))_{\Gamma_{int}} \quad \forall w \in V$$

Consider a Poisson's problem with

$$a(w,u) = \int_\Omega (\nabla w \cdot \alpha \nabla u)\, d\Omega, \quad (w,f) = \int_\Omega w \cdot f\, d\Omega$$

$$(w,h)_{\Gamma_h} = \int_{\Gamma_h} (w \cdot h)\, d\Gamma$$

Then

$$a(w,e)_{\Omega_i} = \int_{\Omega_i} \nabla w \cdot \alpha \nabla e\, d\Omega, \quad (w,f) = \int_{\Omega_i} f \cdot w\, d\Omega$$

$$(w,h)_{\Gamma_{hi}} = \int_{\Gamma \cap \Gamma_h} (w \cdot h)\, d\Gamma \leftarrow \text{only portion on boundary}$$

$$(w, \mathcal{L}(u))_{\Gamma_{int}} = (w, \alpha u_n)_{\Gamma_{int}} = \int_{\Gamma_i \not\subset \Gamma} w \alpha u_n \, d\Gamma$$

(recall $w \equiv 0$ on Γ_g

$\mathcal{V} = H_0^1$

$u_n \Rightarrow$ normal derivative of u

$\Gamma_i \Rightarrow$ element boundary

To calculate an error estimate we introduce finite dimensional versions of e and w with E and w^*

$$a(w^*, E)_{\Omega_i} = (w^*, f)_{\Omega_i} + (w^*, h)_{\Gamma_{h_i}} - a(w^*, u^h)_{\Omega_i} + (w^*, \alpha u_n)_{\Gamma_i^{int}}$$

$$\forall w^* \in \mathcal{V}^*$$

As stated we are still unable to solve the element problems because of the inclusion of the u_n which of course we do not know

need an approximation for this term.

The simplest approximation

$u_n \approx \frac{1}{2}(u_n^{h+} + u_n^{h-})$ (average of the F.E. solution from the 2 sides)

Under the assumption of nodal super convergence of nodal u_i^h, we now have a solvable element by element procedure.

$$a(w^*, E)_{\Omega_i} = (w^*, f)_{\Omega_i} + (w^*, h)_{\Gamma_{h_i}} - a(w^*, u^h)_{\Omega_i}$$

$$+ (w^*, \tfrac{1}{2}\alpha(u_n^{h+} + u_n^{h-}))_{\Gamma_{int}} \quad \forall w^* \in \mathcal{V}^*$$

Lots of variants of this approach have been developed

Recovery Techniques (Zienkiewicz and Zhu)

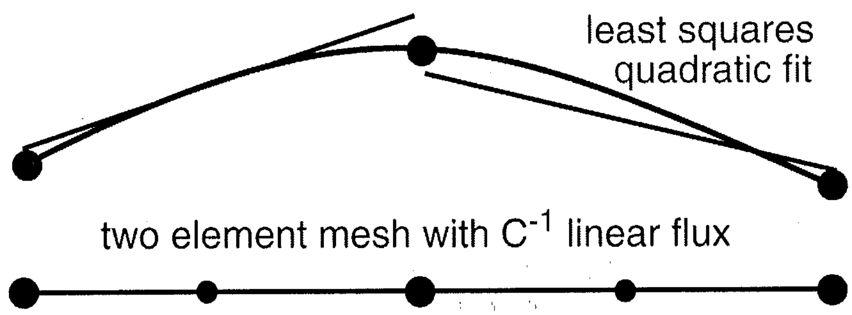
Local construction of a “recovered” smooth flux $\sigma = \alpha\nabla u$ over patches of elements used to define the error in the energy norm in terms of the difference between the FE and recovered flux.

Key ingredients include:

- definition of the patch
- selection of the smooth basis over the patch
- projection method used to fix values of smooth field - typically an integral least squares fit

In specific cases has been shown to yield bounded estimates

In most cases the local contributions useful for adaptive control



Lets define a C^0 stress field

$\sigma^* = N\hat{\sigma}$ ← discrete (nodal) values

Where N is a set of C^0 shape functions –
N could be same as used for u^h or some other C^0 shape functions (typically of equal or lower order than those used for shape functions

Since we trust the accuracy of the energy integral could look to solve for $\hat{\sigma}$ by minimizing – the energy difference

$$\min \int_\Omega (\sigma^* - \sigma^h)\, D^{-1} (\sigma^* - \sigma^h)\, d\Omega$$

to simplify a bit simply assume $D = I$ –
then we have

$$\min \int_\Omega (\sigma^* - \sigma^h)^2 d\Omega = \min \int_\Omega (N\hat{\sigma} - \sigma^h)^2 d\Omega -$$

now this is a function of $\hat{\sigma}$ only (only parameters)
The min corresponds to a stationary point

$$\frac{\partial \left(\int_\Omega (N\hat{\sigma} - \sigma^h)^2 d\Omega \right)}{\partial \hat{\sigma}} = 0 = 2 \int_\Omega N^T (N\hat{\sigma} - \sigma^h)\, d\Omega$$

$$\text{or} \quad \int_\Omega N^T N \, d\Omega \, \hat{\sigma} = \int_\Omega N^T \sigma^h d\Omega$$

$$\hat{\sigma} = \left(\int_\Omega N^T N \, d\Omega \right)^{-1} \int_\Omega N^T \sigma^h d\Omega$$

This would be expensive to solve -

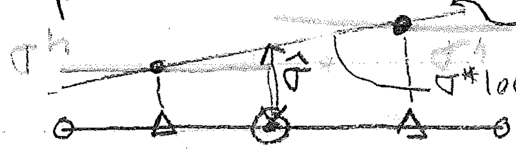
One Simple idea is to cheat a bit on $\left(\int_\Omega N^T N d\Omega\right)^{-1}$ and use a "lumped mass" idea as often done is structural dynamics - This gives a diag. matrix for $\int_\Omega N^T N d\Omega$ which is then trivial to solve.

Not really an ideal solution

An alternative is some more "local" way to get terms in $\hat{\sigma}$.
Assuming for the moment $\hat{\sigma}$ are nodal values
Idea is to use "accurate" values of σ^h in elements $\hat{\sigma}$ bounds to determine the nodal values of $\hat{\sigma}$ (one at a time)

Recovery using local discrete patches and least squares
ZZ - IJNME - 91 or 92

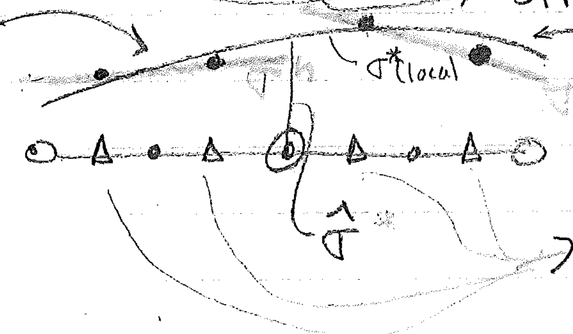
Solve for nodal $\hat{\underset{\sim}{\sigma}}$ values by using a polynomial expansion $\underset{\sim}{\sigma}_p^*$ over the patch of elements using that node



$\underset{\sim}{\sigma}_p^* = P\underset{\sim}{a}$ ⇐ Linear here

→ stresses at "optimal" points

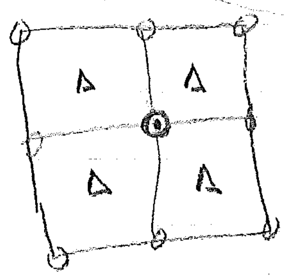
Least squares fit



$\underset{\sim}{\sigma}_p^* = \underset{\sim}{P}\,\underset{\sim}{a}$ ⇐ quadratic here

stresses at "optimal" pts

⇒ Solve for σ_p^* and take the values it gives at the node for $\hat{\sigma}$ for that node for $\underset{\sim}{\sigma}^* = N\hat{\sigma}$



← patch

1-D

$\underset{\sim}{P} = \lfloor 1, x, x^2 \ldots x^p \rfloor$, $\underset{\sim}{a} = \lfloor a_1, a_2 \ldots a_{p+1} \rfloor^T$

2-D Linear $\underset{\sim}{P} = \lfloor 1, x, y \rfloor$, $\underset{\sim}{a} = \lfloor a_1, a_2, a_3 \rfloor^T$

Quad. $\underset{\sim}{P} = \lfloor 1, x, y, x^2, xy, y^2 \rfloor$ $\underset{\sim}{a} = \lfloor a_1, a_2, \ldots a_6 \rfloor^T$

end 11/21

Fit ~~to~~ of $\underset{\sim}{P}$ to "optimal" stress points -

Discrete -

$$\text{Min}\,F(a) = \sum_{i=1}^{n} \left(\sigma_p^*(x_i, y_i) - \sigma^h(x_i, y_i) \right)^2$$

$$\text{Min}\,F(a) = \sum_{i=1}^{n} \left(\underset{\sim}{P}(x_i, y_i)\underset{\sim}{a} - \underset{\sim}{\sigma}^h(x_i, y_i) \right)^2$$

The min implies -

$$\frac{\partial F(a)}{\partial \underset{\sim}{a}} = 0 = \sum_{i=1}^{n} \underset{\sim}{P}^{T}(x_i y_i)\underset{\sim}{P}(x_i y_i)a - \sum_{i=1}^{n} \underset{\sim}{P}^{T}(x_i y_i)\underset{\sim}{\sigma}^{h}(x_i y_i)$$

or $\Rightarrow$ $\underset{\sim}{a} = \underset{\sim}{A}^{-1}\underset{\sim}{b}$

$$\underset{\sim}{A} = \sum_{i=1}^{n} P^{T}(x_i y_i)\, P(x_i y_i) \quad , \quad \underset{\sim}{b} = \sum_{i=1}^{n} P^{T}(x_i y_i)\,\underset{\sim}{\sigma}^{h}(x_i y_i)$$

As Local L_2 projection

$$\text{Min } F(a) = \int_{\Omega_s} (\underset{\sim}{\sigma}_p^{*} - \underset{\sim}{\sigma}^{h})^2 \, d\Omega = \int_{\Omega_s} (\underset{\sim}{P}\underset{\sim}{a} - \underset{\sim}{\sigma}^{h})^2 \, d\Omega .$$

$\uparrow$ patch

$$\frac{\partial F(a)}{\partial \underset{\sim}{a}} = 0 = \int_{\Omega_s} (\underset{\sim}{P}^{T} P a - \underset{\sim}{P}^{T} \underset{\sim}{\sigma}^{h}) \, d\Omega$$

$$\underset{\sim}{a} = \underset{\sim}{A}^{-1}\underset{\sim}{b} \quad , \quad A = \int_{\Omega_s} P^{T} \underset{\sim}{P} \, d\Omega \quad , \quad b = \int_{\Omega_s} \underset{\sim}{P}^{T} \underset{\sim}{\sigma}^{h} \, d\Omega$$

Recently an improvement to this approach in which equilibrium conditions are added to the process -

equilibrium in terms of flux (stress) quantities

$$\mathcal{D}\underset{\sim}{\sigma} + \underset{\sim}{b} = 0 \quad \text{in } \Omega$$

natural b.c.

$$\mathcal{D}^{1}\sigma - \underset{\sim}{h} = 0 \quad \text{on } \Gamma_h$$

$$\text{Min } \tilde{F}_a = F_a + \phi_1 \int_{\Omega_s} (\mathcal{D}\sigma_p^{*} + \underset{\sim}{b})^2 \, d\Omega + \phi_2 \int_{\Gamma_{hs}} (\mathcal{D}^{1}\sigma_p^{*} - \underset{\sim}{h})^2 \, d\Gamma$$

$$\frac{\partial \tilde{F}_a}{\partial a} = 0 = \int_{\Omega_s} \underset{\sim}{P}^T \underset{\sim}{P}\, d\Omega\, \underset{\sim}{a} - \int_{\Omega_s} \underset{\sim}{P}^T \underset{\sim}{\sigma}^h\, d\Omega + \phi_1 \int_{\Omega} \mathcal{D}\underset{\sim}{P}^T \mathcal{D}\underset{\sim}{P}\, d\Omega\, a$$

$$+ \phi_1 \int_{\Omega_s} \mathcal{D}\underset{\sim}{P}^T \underset{\sim}{b}\, d\Omega + \phi_2 \int_{\Gamma_{hs}} \mathcal{D}'\underset{\sim}{P}^T \mathcal{D}'P\, d\Gamma\, a - \phi_2 \int_{\Gamma_{hs}} \mathcal{D}'\underset{\sim}{P}^T \underset{\sim}{h}\, d\Gamma$$

$$\left\{ \int_{\Omega_s} \underset{\sim}{P}^T \underset{\sim}{P}\, d\Omega + \phi_1 \int_{\Gamma_s} \mathcal{D}\underset{\sim}{P}^T \mathcal{D}P\, d\Omega + \phi_2 \int_{\Gamma_{hs}} \mathcal{D}'\underset{\sim}{P}^T \mathcal{D}'P\, d\Gamma \right\} a =$$

$$\int_{\Omega_s} \underset{\sim}{P}^T \underset{\sim}{\sigma}^h\, d\Omega - \phi_1 \int_{\Omega} \mathcal{D}\underset{\sim}{P}^T \underset{\sim}{b}\, d\Omega + \phi_2 \int_{\Gamma_{hs}} \mathcal{D}'\underset{\sim}{P}^T \underset{\sim}{h}\, d\Gamma$$

There are a number of implementation issues associated with selection of shape functions, dealing with b'dry nodes, etc. that require consideration - see Refs.

The various error estimation techniques employ ~~various~~ specific steps of approximation, either in their basic formulation or in making them computationally efficient. A common approx. used is to assume that some finite element soln quantity is super convergent (at least more accurate) at specific points.

A basic proof of super convergence is to show a higher order of convergence at the superconvergent points - This is quite difficult and has only been done for specific situations. - They do not extend to many practical cases.

Babuska, et al. recently proposed an alternative is to examine the solution is always better than at the worst point in the patch.

$$\frac{|F(u(\bar{x})) - F(u_h(\bar{x}))|}{\max\limits_{x \in \omega_h} |F(u(x)) - F(u_h(x))|} \le n \quad \text{as } h \to 0$$

↑ "patch of interest"

0% superconvergent

if $n = 0$ then

$$|F(u(\bar{x})) - F(u_h(\bar{x}))| \approx Ch^{k+\mu} \qquad \mu > 0$$

while

$$\max_{x \in \omega_h} |F(u(x)) - F(u_h(\bar{x}))| \approx Ch^k$$

if $0 < n < 1$ you can still make use of it, particularly if it "small"

They have devised a method by which they can evaluate the boundaries for various values of n. The results are complex functions of mesh topology, etc. and therefore must be calculated with complex procedures –

They have shown for specific situations n=0 points and n>0 regions for various situations.

Goal - Oriented Error Estimates

- Instead of looking at the error due to discretization of the PDE, one can consider the error in a specific quantity of interest. This is often called goal-oriented or adjoint based error estimation.
- This quantity of interest is the functional, $J(\cdot)$, which produces a scalar output,

$$J(u): \mathcal{V} \rightarrow \mathbb{R}.$$

  - Ex : Pressure difference on a surface S,

$$J(u) = \frac{1}{2} \int_S (p - p_{target})^2 dS\,.$$

- The error in the functional can be well approximated with the residual of the PDE. This is done through the use of the **adjoint,** ψ.

- Continuous Variational Problem Statement:
  - **Primal Problem:** Find $u \in \mathcal{V}$, such that

$$R(u, w) = 0 \;\; \forall\, w \in \mathcal{V} \qquad \text{(EQ 4)}$$

  - **Adjoint (Dual) Problem:** Find $\psi \in \mathcal{V}$, such that

$$R'[u](w, \psi) = J'[u](w) \;\; \forall\, w \in \mathcal{V} \qquad \text{(EQ 5)}$$

  Where $R'[u](v, \psi)$ is a Fréchét linearization with respect to u.

Goal - Oriented Error Estimates

- Using (EQ4) and (EQ5), the error in the functional due to a pertubation in the solution can be evaluated by considering the impact of this solution perturbation on the residual.
- Let u_h be the discrete solution to the primal problem and $u_h{+}\delta u_h$ the perturbed solution. Similarly, let ψ_h be the discrete solution to the adjoint problem. Then the error in the functional, δJ, is given by:

$$\begin{aligned}
\delta J = J(u_h + \delta u_h) - J(u_h) &= J'[u_h](\delta u_h) \\
&= R'[u_h](\delta u_h, \psi_h) \\
&= R(u_h + \delta_h, \psi_h) - R(u_h, \psi_h) \\
&= \delta R(\psi_h)
\end{aligned}$$

$$\delta J = R(u_h + \delta_h, \psi_h) \qquad \text{(EQ 6)}$$

Which is (EQ4), with the choice $w = \psi_h$ and perturbed solution. (EQ 6) gives an error estimate, and shows the error in a chosen functional is directly related to the residual with the trial function ψ_h.

Goal - Oriented Error Estimates

When a basis has been chosen for (EQ4) and (EQ5), a discrete formulation can be used for evaluating the error estimate. Consider a coarse solution u_H and a fine solution, u_h. Additionally let $u_H^h \equiv I_h^H u_h$ be the representation of the coarse solution on the fine grid, where I_h^H is a prolongation matrix. The error in the functional is given by

$$J(u_H) - J(u_h) = (\psi_h^{mv})^T R(u_h^H) \qquad \text{(EQ 7)}$$

Where ψ_h^{mv} is the solution to the *mean-value adjoint problem*

$$\left(\bar{R}[u_h, u_h^H]\right)^T \psi_h^{mv} = \left(\bar{J}[u_h, u_h^H]\right)^T \qquad \text{(EQ 8)}$$

Where

$$\bar{R}[u_h, u_h^H] = \int_0^1 \frac{\partial R_h}{\partial u_h}[u_h + \theta(u_h^H - u_h)]\, d\theta$$

$$\bar{J}[u_h, u_h^H] = \int_0^1 \frac{\partial J_h}{\partial u_h}[u_h + \theta(u_h^H - u_h)]\, d\theta$$

Further, since we do not want to solve (EQ 8) on the fine grid, it is common to use $\psi_h^{mv} = \psi_h^{H,mv} - \delta\psi_h^{mv}$, which means (EQ 7) can be written as

$$J(u_H) - J(u_h) = \underbrace{\left(\psi_h^{H,mv}\right)^T R(u_h^H)}_{computable\ correction} - \underbrace{(\delta\psi_h^{mv})^T R(u_h^H)}_{remaining\ error}$$

Goal - Oriented Error Estimates

References:

- K. Fidkowski and D. Darmofal, *Review of Output-Based Error Estimation and Mesh Adaptation in Computational Fluid Dynamics.* AIAA Conferece proceedings, Vol 49, No 4, April 2011.
- R. Becker and R. Rannacher, *An optimal control approach to a posteriori error estimation in finite element methods.* Acta Numerica 2001.
- N. Pierce and M.B. Giles, *Adjoint Recovery of Superconvergent Functionals from PDE Approximations.* Siam Review, Vol 42. No. 2, p. 247-264, 2000.

EXTRACTION TECHNIQUE FOR BOUNDARY STRESSES AND DISPLACEMENTS

Need accurate values of stress concentrations that occur at boundaries

FE solution for boundary stresses is poor

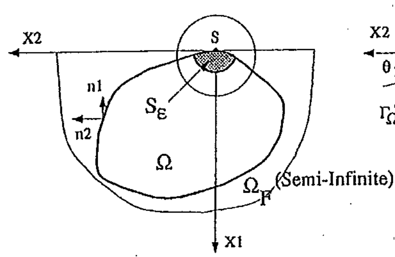
Adaptive control of energy or global L_2 norm does not ensure accuracy of local quantities

Extraction techniques can provide accurate pointwise values

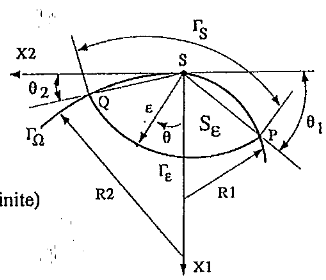
Construction of appropriate integral equations is central to extraction process

ref. Q. Niu and M.S. Shephard, "Super convergent Extraction Techniques for Finite Element Analysis", IJNME, Vol. 36, pp. 811-836, 1993

THE BOUNDARY EXTRACTION MODEL



The Orientation of Ω and Ω_F



The Extraction Model at S

The physical Problem Domain & the Boundary Extraction Model

The Boundary Extraction Location has to be:

- Traction free.
- On a 'smooth' boundary segment of the domain.

1. Extraction of Boundary Displacements

Let u_i^* — virtual displacements and σ_{ij}^* — virtual stresses. The principle of virtual displacements for problem (1) states :

equilibrium in strong form   perscribed   perscribed

$$\int_{\Omega} \sigma_{ij,j} u_i^* \, d\Omega = \int_{\Gamma_\sigma} (p_i - \bar{p}_i) u_i^* d\Gamma + \int_{\Gamma_u} (\bar{u}_i - u_i) p_i^* d\Gamma \tag{4}$$

$p_i^* = \sigma_{ij}^* n_j$ — virtual tractions on the boundary.

normal

Integrating by parts twice,

$\sigma_{ij}^* n_j$ $\sigma_{ij} n_j$

$$\int_{\Omega} \sigma_{ij,j}^* u_i \, d\Omega = \int_{\Gamma_\Omega} u_i p_i^* d\Gamma - \int_{\Gamma_\Omega} p_i u_i^* d\Gamma \tag{5}$$

$$\Gamma_\Omega = \Gamma_u \cup \Gamma_\sigma$$

(0,0)

To extract displacements at *S*, choose u_i^* such that

$$\sigma_{ij,j}^* \pm \delta_k = 0 \tag{6}$$

use a fundamental solution for $\sigma_{ij,j}$

in the semi-infinite plane.

δ_k — unit load at (0,0) acting in the k direction. Green's function

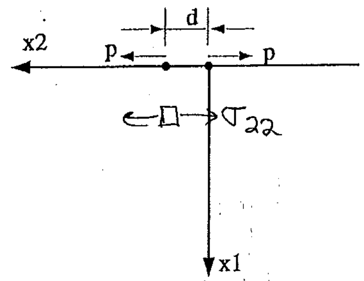
then,

$$u_k(0,0) = \int_{\Gamma_\Omega} p_i u_i^* d\Gamma - \int_{\Gamma_\Omega} u_i p_i^* d\Gamma \tag{7}$$

u_i^*, p_i^* — displacements and tractions due to δ_k.

2. Extraction of Boundary Stresses

Use a different fundamental solution: di-pole solution in the semi-infinite plane.



A Di-pole Model is used as fundamental solutions for stress extraction

To extract σ_{22} at $S(0,0)$, the fundamental solutions

others are known b'dry tractions -

are (plane stress):

$$
\begin{aligned}
u_1^* &= -\frac{2(1+\nu)\cos^3\theta - 2\nu\cos\theta}{E\pi r}\\
u_2^* &= -\frac{2\sin\theta\left(\sin^2\theta + \nu\cos^2\theta\right)}{E\pi r}\\
\sigma_{11}^* &= \frac{2\cos^2\theta\left(4\cos^2\theta - 3\right)}{\pi r^2}\\
\sigma_{22}^* &= -\frac{2\left(1 - 5\cos^2\theta + 4\cos^4\theta\right)}{\pi r^2}\\
\sigma_{12}^* &= \frac{\sin 4\theta}{\pi r^2}
\end{aligned}
\qquad (12)
$$

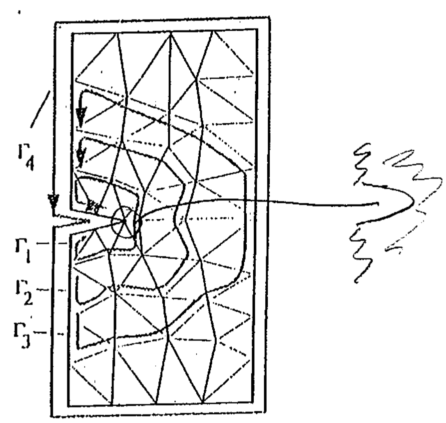
Same operations yield

$$\sigma_{22}(0,0) = \frac{R_1 - R_2}{\pi R_1 R_2} E\, u_2(0,0) + E\left(\int_{\Gamma_\Omega} u_i^* p_i \, d\Gamma - \int_{\Gamma_\Omega} p_i^* u_i \, d\Gamma \right) \quad (13)$$

the extraction form is

$$\sigma_{22}^{(\mathrm{extract})}(0,0) = \frac{R_1 - R_2}{\pi R_1 R_2} E\, U_2(0,0) + E\left(\int_{\Gamma_\Omega} P_i u_i^{D*} d\Gamma - \int_{\Gamma_\Omega} U_i p_i^{D*} d\Gamma \right) \quad (14)$$

Integration paths sensitivity study:

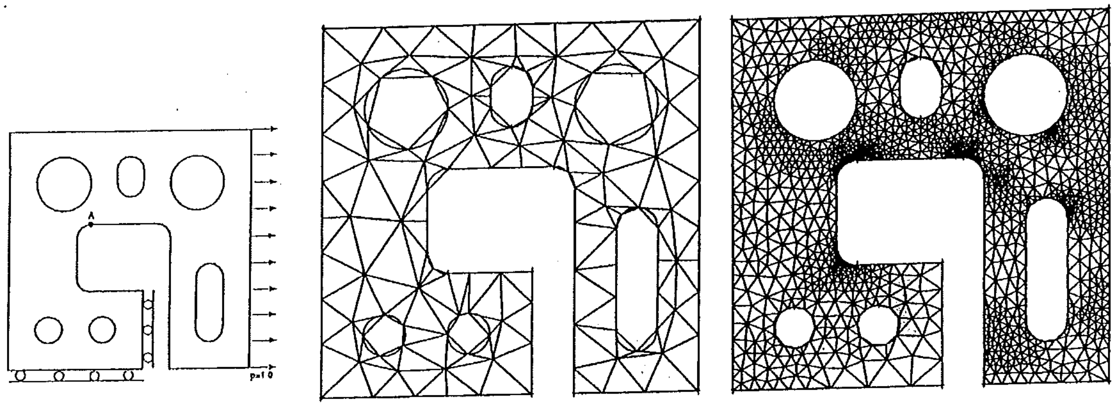


← not a sharp crack- a rounded one

Test Problem 3: The Mesh and Different Integration Paths Used for Boundary Stress Extraction

	Γ_1	Γ_2	Γ_3	Γ_4
Extracted Stress	9.2447555	9.4563938	9.5375545	9.4873897

EXAMPLE WITH LINEAR ELEMENTS



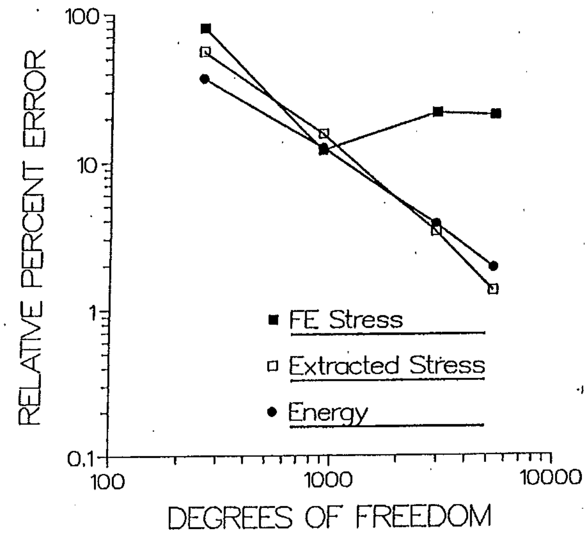
Initial Mesh　　Final Mesh

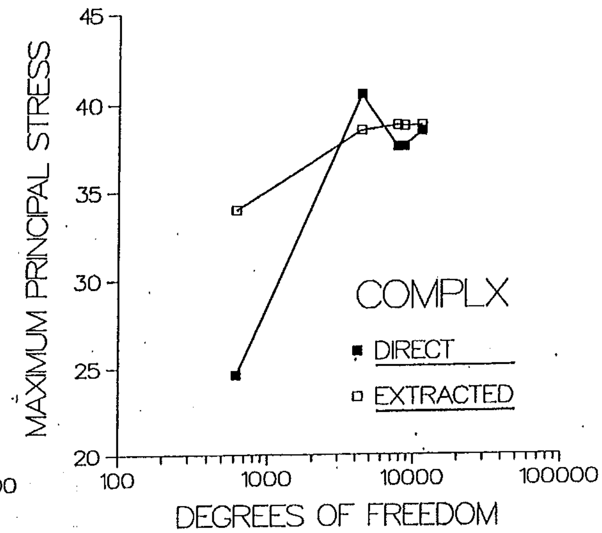
Degrees of Freedom	Direct FE Stress	Extracted Stress	Total Strain Energy
256	7.70	17.15	3500.6
902	34.06	32.66	4859.2
2936	30.32	37.46	5355.4
5314	30.57	38.25	5459.4
Estimated Convergence Rate	0.37	1.21	0.97

EXAMPLE WITH QUADRATIC ELEMENTS

Degrees of Freedom	Direct FE Stress	Extracted Stress	Total Strain Energy
622	24.60	33.95	5128.9
4336	40.50	38.49	5551.1
7714	37.55	38.79	5564.8
8568	37.59	38.74	5565.9
11298	38.46	38.77	5567.3
Estimated Convergence Rate	0.97	2.02	2.13

LINEAR ELEMENTS **QUADRATIC ELEMENTS**





CORRECTION INDICATION

Results of the current analysis used to indicate where and how to improve the discretization

Common to use the elemental error estimator and improve where it is higher than a prescribed value

Can use the magnitude of the elemental error plus convergence rate to indicate how much refinement

More advanced correction indicators are needed when there are choices on how to improve like hp-refinement, directional refinement, etc.

Effective correction indication can greatly reduce total solution cost