Momentum-Conserving Velocity

Dan Ibanez

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1 First Principles

Ultimately, what we want to do is conserve the integral of momentum over the domain Ω of a cavity. The momentum is defined as the volume integral of the product of velocity and density, and we have some change happening that will alter both fields. The *donor* fields are those before the change, and the *target* fields are those after the change. Thus our highest-level objective is the following:

$$\int_\Omega \rho_T g \,\mathrm{d}V = \int_\Omega \rho_D p \,\mathrm{d}V \tag{1}$$

Ω	cavity domain
ρ_T	target density field
ρ_D	donor density field
g	target velocity field
p	donor velocity field

Then, in order to control the distribution of momentum throughout the domain, we can use a weighted residual formulation. By the time it gets discretized, this weighted residual will enforce an L_2 minimization of the difference between the momentum fields.

$$\forall w \in H_0^1 : \int_\Omega w\rho_T g \,\mathrm{d}V = \int_\Omega w\rho_D p \,\mathrm{d}V \tag{2}$$

w	weighting function
H_0^1	space of all weighting functions

We also assume that ρ_T has been given by a prior process, so our goal is to find $v_T \in H_T^1$ such that Equation 2 is satisfied. We define the function spaces H_T^1 and H_0^1 in order to also satisfy Dirichlet boundary conditions on velocity over the whole cavity boundary:

$$(g \in H_T^1) \to (\forall \mathbf{x} \in \partial\Omega : g(\mathbf{x}) = c(\mathbf{x})) \tag{3}$$

$$(w \in H_0^1) \to (w \in H_T^1, \quad \forall \mathbf{x} \in \partial\Omega : w(\mathbf{x}) = 0) \tag{4}$$

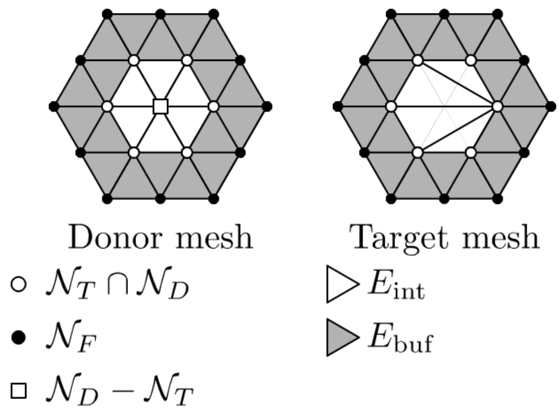

Figure 1: Cavity with buffer layer and associated notation

c	Dirichlet velocity condition given over $\partial\Omega$

2 Discretization

We now discretize the problem by requiring all functions involved to belong to one of six spaces, which are further restricted from their earlier descriptions. Please see Figure 1 for an illustration of the different sets of nodes.

$$
\begin{aligned}
(w \in H_0^1) &= \sum_{I \in \mathcal{N}_T} w_I \varphi_I \\
(g \in H_T^1) &= \sum_{I \in \mathcal{N}_T} g_I \varphi_I + \sum_{I \in \mathcal{N}_F} c(\mathbf{x}_I) \varphi_I \\
(\rho_T \in H_T^0) &= \sum_{e \in E_T} \rho_{Te} \vartheta_e \\
(p \in H_D^1) &= \sum_{I \in \mathcal{N}_D} p_I \phi_I + \sum_{I \in \mathcal{N}_F} c(\mathbf{x}_I) \phi_I \\
(\rho_D \in H_D^0) &= \sum_{e \in E_D} \rho_{De} \theta_e
\end{aligned}
\tag{5}
$$

$\mathcal{N}_T$	set of interior target nodes
$\mathcal{N}_D$	set of interior donor nodes
$\mathcal{N}_F$	set of fixed nodes
E_T	set of target elements
E_D	set of donor elements
φ_I	target basis function of node I
ϕ_I	donor basis function of node I
ϑ_e	target basis function of element e
θ_e	donor basis function of element e
g_I	target velocity value of node I
ρ_{Te}	target density value of element e
$\mathbf{x}_I$	location of node I

Applying these discretizations to Equation 2 yields the following:

$$\begin{aligned}\forall w \in H_0^1 : \int_\Omega \Bigg(\Bigg(\sum_{I\in\mathcal{N}_T} w_I \varphi_I \Bigg) \Bigg(\sum_{e\in E_T} \rho_{Te} \vartheta_e \Bigg) \\ \Bigg(\sum_{I\in\mathcal{N}_T} g_I \varphi_I + \sum_{I\in\mathcal{N}_F} c(\mathbf{x}_I)\varphi_I \Bigg) \,\mathrm{d}V \Bigg) = \\ \int_\Omega \Bigg(\Bigg(\sum_{I\in\mathcal{N}_T} w_I \varphi_I \Bigg) \Bigg(\sum_{e\in E_D} \rho_{De} \theta_e \Bigg) \\ \Bigg(\sum_{I\in\mathcal{N}_D} p_I \phi_I + \sum_{I\in\mathcal{N}_F} c(\mathbf{x}_I)\phi_I \Bigg) \,\mathrm{d}V \Bigg)\end{aligned} \tag{6}$$

We can move the sum over weighting degrees of freedom out of the integrals:

$$\begin{aligned}\forall w \in H_0^1 : \sum_{I\in\mathcal{N}_T} w_I \int_\Omega \Bigg((\varphi_I) \Bigg(\sum_{e\in E_T} \rho_{Te} \vartheta_e \Bigg) \\ \Bigg(\sum_{I\in\mathcal{N}_T} g_I \varphi_I + \sum_{I\in\mathcal{N}_F} c(\mathbf{x}_I)\varphi_I \Bigg) \,\mathrm{d}V \Bigg) = \\ \sum_{I\in\mathcal{N}_T} w_I \int_\Omega \Bigg((\varphi_I) \Bigg(\sum_{e\in E_D} \rho_{De} \theta_e \Bigg) \\ \Bigg(\sum_{I\in\mathcal{N}_D} p_I \phi_I + \sum_{I\in\mathcal{N}_F} c(\mathbf{x}_I)\phi_I \Bigg) \,\mathrm{d}V \Bigg)\end{aligned} \tag{7}$$

Then choose a set of $|\mathcal{N}_T|$ weighting functions such that each one sets $w_I = \delta_{IJ}$ for a different node J. The result is $|\mathcal{N}_T|$ equations for the unknowns g_J:

$$\forall I \in \mathcal{N}_T : \int_\Omega \left(\varphi_I \left(\sum_{e \in E_T} \rho_{Te} \vartheta_e \right) \left(\sum_{J \in \mathcal{N}_T} g_J \varphi_J + \sum_{J \in \mathcal{N}_F} c(\mathbf{x}_J) \varphi_J \right) \mathrm{d}V \right) = \int_\Omega \left(\varphi_I \left(\sum_{e \in E_D} \rho_{De} \theta_e \right) \left(\sum_{J \in \mathcal{N}_D} p_J \phi_J + \sum_{J \in \mathcal{N}_F} c(\mathbf{x}_J) \phi_J \right) \mathrm{d}V \right) \quad (8)$$

Let us also separate each side according to the Dirichlet velocity conditions:

$$\forall I \in \mathcal{N}_T : \int_\Omega \left(\varphi_I \left(\sum_{e \in E_T} \rho_{Te} \vartheta_e \right) \left(\sum_{J \in \mathcal{N}_T} g_J \varphi_J \right) \mathrm{d}V \right) + \int_\Omega \left(\varphi_I \left(\sum_{e \in E_T} \rho_{Te} \vartheta_e \right) \left(\sum_{J \in \mathcal{N}_F} c(\mathbf{x}_J) \varphi_J \right) \mathrm{d}V \right) = \int_\Omega \left(\varphi_I \left(\sum_{e \in E_D} \rho_{De} \theta_e \right) \left(\sum_{J \in \mathcal{N}_D} p_J \phi_J \right) \mathrm{d}V \right) + \int_\Omega \left(\varphi_I \left(\sum_{e \in E_D} \rho_{De} \theta_e \right) \left(\sum_{J \in \mathcal{N}_F} c(\mathbf{x}_J) \phi_J \right) \mathrm{d}V \right) \quad (9)$$

Now note from Figure 1 that the only donor elements $\{e\}$ for which $\int_\Omega \theta_e \phi_J \neq 0$ at any point for any fixed node J is the set of unchanging buffer elements E_{buf}, and that this is the same set of target elements for which $\vartheta_e \phi_J \neq 0$ at any point for any fixed node J. Since these elements are not changing, their basis functions are the same ($\vartheta_e = \theta_e$). Since the basis functions ϕ_J and φ_J of a fixed node J only cover buffer elements, then they too are equal for all fixed nodes. Furthermore, we assume that the transfer algorithm for density leaves the density values of buffer elements unchanged. In total, this means that Dirichlet contributions to momentum are equal for the donor and target meshes:

$$\forall I \in \mathcal{N}_T : \int_\Omega \left(\varphi_I \left(\sum_{e \in E_T} \rho_{Te} \vartheta_e \right) \left(\sum_{J \in \mathcal{N}_F} c(\mathbf{x}_J) \varphi_J \right) \mathrm{d}V \right) = \int_\Omega \left(\varphi_I \left(\sum_{e \in E_D} \rho_{De} \theta_e \right) \left(\sum_{J \in \mathcal{N}_F} c(\mathbf{x}_J) \phi_J \right) \mathrm{d}V \right) \quad (10)$$

It also means that the remaining momentum contributions (from non-fixed nodes) must be made equal for conservation:

$$
\begin{aligned}
\forall I \in \mathcal{N}_T : \int_\Omega \left(\varphi_I \left(\sum_{e \in E_T} \rho_{Te} \vartheta_e \right) \left(\sum_{J \in \mathcal{N}_T} g_J \varphi_J \right) \mathrm{d}V \right) = \\
\int_\Omega \left(\varphi_I \left(\sum_{e \in E_D} \rho_{De} \theta_e \right) \left(\sum_{J \in \mathcal{N}_D} p_J \phi_J \right) \mathrm{d}V \right)
\end{aligned} \tag{11}
$$

We can also factor out the remaining sums over elements and nodes:

$$
\begin{aligned}
\forall I \in \mathcal{N}_T : \sum_{J \in \mathcal{N}_T} g_J \sum_{e \in E_T} \rho_{Te} \int_\Omega \varphi_I \vartheta_e \varphi_J \,\mathrm{d}V = \\
\sum_{J \in \mathcal{N}_D} p_J \sum_{e \in E_D} \rho_{De} \int_\Omega \varphi_I \theta_e \phi_J \,\mathrm{d}V
\end{aligned} \tag{12}
$$

Finally, if we assume that element shape functions are one in the element and zero elsewhere:

$$
\theta_e(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in \Omega_e \\ 0 & \mathbf{x} \notin \Omega_e \end{cases} \tag{13}
$$

Ω_e	domain of element e

Then we can rewrite Equation 12 as:

$$
\begin{aligned}
\forall I \in \mathcal{N}_T : \sum_{J \in \mathcal{N}_T} g_J \sum_{e \in E_T} \rho_{Te} \int_{\Omega_e} \varphi_I \varphi_J \,\mathrm{d}V = \\
\sum_{J \in \mathcal{N}_D} p_J \sum_{e \in E_D} \rho_{De} \int_{\Omega_e} \varphi_I \phi_J \,\mathrm{d}V
\end{aligned} \tag{14}
$$

Equation 14 leads directly to a linear system $\mathbf{M}\mathbf{g} = \mathbf{b}$ if one considers the degrees of freedom to be the interior target velocities g_J:

$$
\mathbf{M}_{IJ} = \sum_{e \in E_T} \rho_{Te} \int_{\Omega_e} \varphi_I \varphi_J \,\mathrm{d}V \tag{15}
$$

$$
\mathbf{b}_I = \sum_{J \in \mathcal{N}_D} p_J \sum_{e \in E_D} \rho_{De} \int_{\Omega_e} \varphi_I \phi_J \,\mathrm{d}V \tag{16}
$$

3 Integration

First, we will separate Equation 16 into terms involving interior and buffer elements of the donor mesh:

$$\mathbf{b}_I = \sum_{J \in \mathcal{N}_D} p_J \left(\sum_{e \in E_{D\text{int}}} \rho_{De} \int_{\Omega_e} \varphi_I \phi_J \, \mathrm{d}V + \sum_{e \in E_{\text{buf}}} \rho_{De} \int_{\Omega_e} \varphi_I \phi_J \, \mathrm{d}V \right) \tag{17}$$

Since buffer elements are the same in both meshes, we know that for a given buffer element e and adjacent node I that $(\varphi_I = \phi_I)$ over the domain Ω_e. This simplifies the second term of Equation 17:

$$\mathbf{b}_I = \sum_{J \in \mathcal{N}_D} p_J \left(\sum_{e \in E_{D\text{int}}} \rho_{De} \int_{\Omega_e} \varphi_I \phi_J \, \mathrm{d}V + \sum_{e \in E_{\text{buf}}} \rho_{De} \int_{\Omega_e} \varphi_I \varphi_J \, \mathrm{d}V \right) \tag{18}$$

So now, for both Equations 15 and 18, we will have to integrate this term over a target mesh element e, for two target nodes I and J (recall that buffer elements are a subset of target elements):

$$\int_{\Omega_e} \varphi_I \varphi_J \, \mathrm{d}V \tag{19}$$

This we can express as a numerical integration (leaving aside which points are used for the moment):

$$\sum_{p=1}^{n_{\text{IP}}} w_p \varphi_I(\xi_p) \varphi_J(\xi_p) \det(\mathbf{J}_e(\xi_p)) \tag{20}$$

This can be simplified quite a bit because the element is linear. First, the Jacobian $\mathbf{J}_e$ is constant over the element, and can be factored out:

$$\det(\mathbf{J}_e) \sum_{p=1}^{n_{\text{IP}}} w_p \varphi_I(\xi_p) \varphi_K(\xi_p) \tag{21}$$

Second, the global basis functions φ_I and φ_J restricted to the element domain Ω_e are just a pair of local element basis functions N_i and N_j where $i \neq j$. That means the integral of their product in parametric space is a constant independent of the element coordinates. Furthermore, due to the symmetry of tetrahedra, it is the same constant for all i and j. Using the 2nd order accurate 4-point quadrature rule for tetrahedra (and more generally the $(d+1)$-point rule for a d-dimension simplex), we find the following identity:

$$\sum_{p=1}^{d+1} \left(\frac{V_0}{d+1} \right) N_i(\xi_p) N_j(\xi_p) = \frac{V_0}{(d+1)(d+2)}, \quad i \neq j \tag{22}$$

The weights w_p are all equal to $V_0/(d+1)$, where V_0 is the volume of the parent parametric space element, so that value was factored out. Assuming this holds true, and because the quadrature rule used is exact for the polynomial being integrated, then we have the following fully accurate substitution for this particular integral over a simplex:

$$\int_{\Omega_e} \varphi_I \varphi_K \,\mathrm{d}V = \frac{\det(\mathbf{J}_e)V_0}{(d+1)(d+2)} = \frac{V_e}{(d+1)(d+2)} \tag{23}$$

Where V_e is the volume (size) of simplex e. Substituting Equation 23 into Equation 15 yields the following:

$$\mathbf{M}_{IJ} = \frac{1}{(d+1)(d+2)} \sum_{e \in E_T} \rho_{Te} V_e \tag{24}$$

And similarly for Equation 18:

$$\mathbf{b}_I = \sum_{J \in \mathcal{N}_D} p_J \left(\sum_{e \in E_{D\mathrm{int}}} \rho_{De} \int_{\Omega_e} \varphi_I \phi_J \,\mathrm{d}V + \frac{1}{(d+1)(d+2)} \sum_{e \in E_{\mathrm{buf}}} \rho_{De} V_e \right) \tag{25}$$

This leaves us with the more difficult integral:

$$\int_{\Omega_e} \varphi_I \phi_J \,\mathrm{d}V, \quad e \in E_{D\mathrm{int}}, \quad I \in \mathcal{N}_T, \quad J \in \mathcal{N}_D \tag{26}$$

The trouble in Equation 26 is that φ_I is only fully differentiable in the domains of target elements, not the donor domains being integrated over. This is why we use intersections of the interior donor and target elements (recall from Figure 1 that the domain covered by interior elements is the same is both meshes).

$$\int_{\Omega_e} \varphi_I \phi_J \,\mathrm{d}V = \sum_{o \in E_{T\mathrm{int}}} \int_{\Omega_e \cap \Omega_o} \varphi_I \phi_J \,\mathrm{d}V \tag{27}$$

We will use R3D's polyhedral intersection capability to compute $\Omega_e \cap \Omega_o$, and its polynomial integration capability to integrate the polynomial $\varphi_I \phi_J$ over the intersection polyhedron. φ_I can be expressed as a continuous linear polynomial $f(x, y, z)$ inside Ω_o, likewise for ϕ_J inside Ω_e.