Momentum-Conserving Velocity Counter-Example

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1 1D Edge Collapse

This is a 1D edge collapse where the element size (node spacing) is all unity before the edge collapse, and likewise we assume the density is one everywhere in both donor and target meshes. See Figure 1 for the situation.

Just looking at basic conservation, the momentum of the donor and target cavities are:

$$\mathbf{p}_D = \frac{1}{2}(p_l + 2p_a + 2p_b + 2p_c + p_r) \tag{1}$$

$$\mathbf{p}_T = \frac{1}{2}(p_l + 3g_a + 3g_c + p_r) \tag{2}$$

Setting them equal to each other cancels the fixed node contributions on either side, and leaves us with this conservation goal to be satisfied at minimum:

$$\frac{3}{2}(g_a + g_c) = p_a + p_b + p_c \tag{3}$$

The scheme we tried gives a linear system like so:

$$\mathbf{M}\begin{bmatrix} g_a \\ g_c \end{bmatrix} = \mathbf{b} \tag{4}$$

$$\mathbf{M}_{IJ} = \frac{1 + \delta_{IJ}}{(d+1)(d+2)}\mathbf{A}_{IJ} \tag{5}$$

$$\mathbf{A}_{IJ} = \sum_{e \in E_T(I,J)} \rho_{Te} V_e \tag{6}$$

$$\begin{aligned} \mathbf{A}_{11} &= \mathbf{A}_{22} = 1 + 2 = 3 \\ \mathbf{A}_{12} &= \mathbf{A}_{21} = 2 \end{aligned} \tag{7}$$

$$\mathbf{M} = \frac{1}{6}\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \tag{8}$$

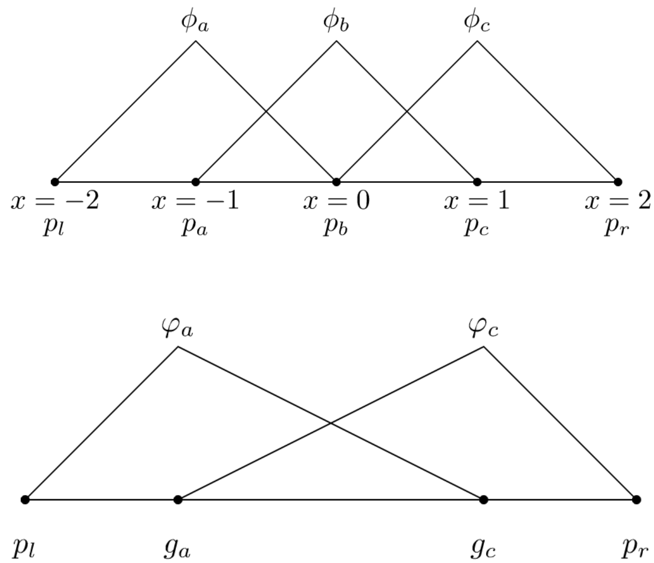

Figure 1: (top) donor cavity (bottom) target cavity

$$\mathbf{M}^{-1} = \frac{3}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \tag{9}$$

$$\mathbf{b}_1 = \frac{1}{3}p_a + p_a\int_{-1}^{0}\varphi_a\phi_a\,\mathrm{d}x + p_b\int_{-1}^{1}\varphi_a\phi_b\,\mathrm{d}x + p_c\int_{0}^{1}\varphi_a\phi_c\,\mathrm{d}x \tag{10}$$

$$\mathbf{b}_2 = \frac{1}{3}p_c + p_c\int_{0}^{1}\varphi_c\phi_c\,\mathrm{d}x + p_b\int_{-1}^{1}\varphi_c\phi_b\,\mathrm{d}x + p_a\int_{-1}^{0}\varphi_c\phi_a\,\mathrm{d}x \tag{11}$$

Evaluating the integrals of basis function pairs:

$$\int_{-1}^{0}\varphi_a\phi_a\,\mathrm{d}x = \int_{0}^{1}\varphi_c\phi_c\,\mathrm{d}x = \int_{0}^{1}(1-\tfrac{1}{2}x)(1-x)\,\mathrm{d}x = \frac{5}{12} \tag{12}$$

$$\int_{-1}^{1}\varphi_a\phi_b\,\mathrm{d}x = \int_{-1}^{1}\varphi_c\phi_b\,\mathrm{d}x = \int_{0}^{1}(1-\tfrac{1}{2}x)x\,\mathrm{d}x + \int_{1}^{2}(1-\tfrac{1}{2}x)(2-x)\,\mathrm{d}x = \frac{1}{3}+\frac{1}{6} = \frac{1}{2} \tag{13}$$

$$\int_{-1}^{0}\varphi_c\phi_a\,\mathrm{d}x = \int_{0}^{1}\varphi_a\phi_c\,\mathrm{d}x = \int_{1}^{2}(1-\tfrac{1}{2}x)(x-1)\,\mathrm{d}x = \frac{1}{12} \tag{14}$$

Substituting integral values back:

$$\mathbf{b}_1 = \frac{1}{3}p_a + \frac{5}{12}p_a + \frac{1}{2}p_b + \frac{1}{12}p_c = \frac{1}{12}(9p_a + 6p_b + p_c) \tag{15}$$

$$\mathbf{b}_2 = \frac{1}{12}(9p_c + 6p_b + p_a) \tag{16}$$

Multiplying by $\mathbf{M}^{-1}$ gives:

$$\begin{aligned} g_a &= \frac{3}{8}\left(\frac{3}{12}(9p_a + 6p_b + p_c) - \frac{1}{12}(9p_c + 6p_b + p_a)\right) \\ &= \frac{3}{8 \cdot 12}(26p_a + 12p_b - 6p_c) \end{aligned} \tag{17}$$

$$g_b = \frac{3}{8 \cdot 12}(26p_c + 12p_b - 6p_a) \tag{18}$$

$$\frac{3}{2}(g_a + g_b) = \frac{1}{48}(20p_a + 24p_b + 20p_c) \neq p_a + p_b + p_c \tag{19}$$